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Stochastic analysis of time-variant nonlinear dynamic systems. Part 1: the Fokker–Planck–Kolmogorov equation approach in stochastic mechanics

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The probabilistic description and analysis of the response of time-invariant nonlinear dynamic systems driven by stochastic processes is usually treated by means of evaluation of statistical moments and cumulants of the response. The background of these methods is the Fokker–Planck–Kolmogorov (FPK) equation for a probability density function or the Pugachev equation for a characteristic function, respectively. The exact solutions of these equations are obtained only for isolated cases. For engineering probabilistic analysis of a complex nonlinear systems, different mixed (hybrid) methods in these cases are used. In this study a 'benchmark' solution is obtained on the basis of the FPK equation in conjunction with the method of statistical moments for nonlinear mechanical system with colored parametric excitations. In Part 1 (this part), an exact solution of FPK equation on the basis of asymptotic analysis of nonlinear dynamic behavior of parametric excitation system is discussed. In Parts 2 and 3, applications of this method to stochasticity and stability analysis of nonlinear time-variant systems are considered. A comparison with the accuracy of different statistical methods is discussed. In Parts 4 and 5, a method of stochastic analysis of relativistic and quantum dynamic systems is described on the basis of a generalized stochastic Hamilton–Jacobi equations on a differential manifold as Riemanian geometry. This involves the task of relativistic navigation and dissipative quantum models of a nonlinear parametric oscillator in the presence of stochastic excitations on a differential manifold with different metric tensors of the space–time continuum.

1 INTRODUCTION

The stochastic theory of Markov processes explains many phenomena where fluctuations play a significant role.[1-3] The classical theory of probability and stochastic processes provides powerful tools for the description of classical complex dynamic systems. In accordance with their wide range of applicability, there exist various powerful solution methods for Markov processes based either on the global characterization of the probability evolution by the Onsanger–Machlup path integral or on its local equivalent, the Fokker–Planck–Kolmogorov (FPK) equation.[4,5] In this context, the variables of the system under study are considered as random variables, the time evolution of which is represented by a stochastic process. The FPK equation is the basic evolution equation for a great number of physical and engineering problems. Examples of the former are the small noise expansion and the adiabatic elimination procedure of fast random variables; examples of the latter are the eigenfunction expansion and the continued-fraction method applied to periodically driven systems.[1-5]

This paper, consisting of five parts, deals with two types of nonlinear dynamic systems: one with time-invariant characteristics subjected to parametric excitations; the other with time-variant characteristics. Random change in characteristics is considered as: (1) a stochastic event that does not depend on information about phase coordinates of the dynamic system; and (2) a function of phase coordinates of the dynamic system. A time-invariant dynamic system is obtained as a special case in which a change in

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characteristics has a zero probability for one random realization of motion. Therefore, both types of dynamic systems can be analyzed by common mathematical tools of statistical analysis of parametric systems.

The probabilistic description and analysis of the response of time-invariant nonlinear dynamic systems driven by stochastic processes is usually treated by means of the evaluation of statistical moments and cumulants of the response. The background of these methods is the FPK equation for a probability density function or the Pugachev equation for a characteristic function, respectively. The exact solutions of these equations are obtained only for a separate cases.[1-15]

For engineering probabilistic analysis of complex nonlinear systems, different mixed (hybrid) methods in these cases are used.[1,8,16-28] In this study, a 'benchmark' solution is obtained on the basis of the FPK equation in conjunction with the method of statistical moments for nonlinear mechanical system with colored parametric excitations. In Part 1, an exact solution of the FPK equation on the basis of asymptotic analysis of nonlinear dynamic behavior of parametric excitation system is discussed. In Appendix A, different methods for solving the FPK equation are described. In Parts 2 and 3, applications of this method to stochasticity and stability analysis of nonlinear time-variant systems are considered. In Parts 4 and 5, generalized statistical analysis of quantum and relativistic dynamic systems on a stochastic differential manifold with different metric tensors (as Riemanian and Finsler stochastic geometry of space–time continuum) is carried out on the basis of generalized stochastic Hamilton–Jacobi equations and tensor models of stochastic parallel displacement. A concrete example of stochastic analysis is given as relativistic navigation in the space–time continuum with different metric tensors and damping quantum models of a nonlinear parametric oscillator in present of stochastic excitations are examined.

The models of FPK equations and its applications can be considered from three viewpoints: (1) mathematical modeling of random processes based on Chapman–Kolmogorov equation and solutions of FPK equations; (2) stochastic models of classical (Newton, Lagrange and Hamilton–Jacobi formalisms), relativistic (Einstein) and quantum (nonrelativistic and relativistic) Nelson's mechanics; and (3) stochastic optimal control of dynamic systems.

In this paper, we consider in detail the second problem based on a stochastic Hamilton–Jacobi equations in different space–time continuums.[29,30] Interrelations with the first and third problems in Parts 4 and 5 are discussed in detail. Here we only discuss the main problems of these viewpoints and interrelations with the FPK equations that are used in this paper.

2 THE MODELS OF FPK EQUATIONS IN STOCHASTIC MECHANICS

(1) From the *mathematical viewpoint*, several generic Markov processes are studied when extended to complex measure.[31] For the Markov process, $X(t)$, an indexed family of complex measure is best defined by the conditional structure $F_n(x, t|x_0, t_0, \ldots, x_n, t_n) = Pr\{X(t) \le X|X(t_0) = x_0, \ldots, X(t_n) = x_n\}$, $t_0 \le t_1 \le \ldots t_n \le t$ and $F_n(x, t|x_0, t_0, \ldots x_n) = F_2(x, t|x_n, t_n)$ or for the probability density function $f_n(x, t|x_0, t_0, \ldots x_n, t_n) = f_2(x, t|x_n, t_n)$. If we denote by $F(x, t)$ the probability $Pr\{X(t) \le x\}$ or $f(x,t) = Pr\{x \le x(t) \le x + dx\}$, then we have the Chapman–Kolmogorov relationship:

$$f_2(x, t|x_0, t_0) = \int f_2(x, t|x', t') f_2(x', t'|x_0, t_0)\, dx', \quad \forall t' \in (t_0, t) \tag{1}$$

and

$$f_2(x, t) = \int f_2(x, t|x', t') f_2(x', t')\, dx', \ \forall t' < t \tag{2}$$

Now we put in some constants:

$$a_n(z, \Delta) = \int (x - z)^n f_2(x|z, \Delta)\, dz \tag{3}$$

and for $t_0 = 0$ (assume time homogeneity) from eqn (1) we may write

$$f_2(x|x_0, t + \Delta) = \int f_2(x|z, \Delta) f_2(z|x_0, t)\, dz \tag{4}$$

Assume that a_n/Δ has a nonzero limit for $n = 1, 2$:

$$A(z, t) = \lim_{\Delta \to 0} \frac{a_1(z, \Delta)}{\Delta}, \ B(z, t) = \lim_{\Delta \to 0} \frac{a_2(z, \Delta)}{\Delta} \tag{5}$$

Now use the Chapman–Kolmogorov relationship from eqn (4) and integrating by parts, we obtain:

$$\int \frac{\partial f_2}{\partial t} \varphi(x)\, dx = \int \left\{ \frac{\partial}{\partial z}[A(z) f_2(Z|x_0, t)] - \frac{1}{2} \frac{\partial^2}{\partial z^2}[B(z) f_2(z|x_0, t)] \right\} \times \varphi(z)\, dz \tag{6}$$

where $\varphi(x)$ is a smooth function that tends to zero sufficiently as $x \to \pm\infty$. Since this holds for $\forall \varphi$, we have FPK equation:

$$\frac{\partial f_2(x|x_0, t)}{\partial t} = -\frac{\partial}{\partial x}[f_2(x|x_0, t) A(z, t)] + \frac{1}{2} \frac{\partial^2}{\partial x^2}[f_2(z|x_0, t) B(x, t)] \tag{7}$$

If we assume in FPK eqn (7) that in $B(x) = -(i\hbar)/m$, then we can use the standard transformation:

$$f_2(x|x_0, t) \exp\left\{ -\frac{1}{2} \int_0^x A(\theta, t)\, d\theta \right\} = \psi_2(x|x_0, t) \tag{8}$$

in which case the FPK eqn (7) reduces to

$$i\hbar \frac{\partial \psi_2(x|x_0, t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi_2(x|x_0, t)}{\partial x^2} + V(x, t) \psi_2(x|x_0, t) \tag{9}$$

where

$$V(x,t)=\frac{m}{2}(A(x))^2-i\hbar\frac{dA}{dx}-\frac{i\hbar}{2}\int_0^t\frac{\partial A(\theta,t)}{\partial t}d\theta \qquad (10)$$

This is the Schrödinger equation.

Remark 1. We may write eqn (4) in a more generalized form as:[31]

$$\psi(x_2,t_2)=\int N_2(x_2,t_2|x_1,t_1)\psi(x_1,t_1)\,dx_1 \qquad (11)$$

If take $t_2 = t_1 + \varepsilon$ (ε is small) and choose

$$N_2=\frac{1}{A}\exp\left\{\frac{i}{\hbar}\frac{(x_2-x_1)^2}{\varepsilon}\right\}\exp\left\{\frac{i}{\hbar}\varepsilon V\left(\frac{(x_2+x_1)^2}{2},t\right)\right\},$$

$$A=\left(\frac{2\pi i\hbar\varepsilon}{m}\right)^{1/2},\quad \int N_2\,dx_2=1 \qquad (12)$$

then we have

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{h^2}{2m}\frac{\partial^2\psi}{\partial x^2}+V(x,t)\psi \qquad (13)$$

In this case, N_2 is a genuine complex probability measure and also satisfies an identical equation if we apply this to eqn (1). Eqns (9) and (13) are identical.

These equations, which will be subsumed under the name 'FPK equation', typically have the form:

$$\frac{\partial P(X,t)}{\partial t}=cLP(x,t)\equiv c\left[\frac{1}{2}\frac{\partial^2 D_{ij}(x)}{\partial x_i\partial x_j}-\frac{\partial G_i}{\partial x_i}-V(x)\right]P(x,t) \qquad (14)$$

and are to be solved with some initial condition $P(\mathbf{x}, 0)$. Here, a summation over indices is always implied, if not stated otherwise; L is the FPK operator defined by eqn (14), $\mathbf{x}^T = (x_1,\ldots,x_n)$, and the number c defines the problem under study:[10] for complex $c = i$, $\hbar = 1$ eqn (14) is a Schrödinger equation and for $c = 1$ it is a FPK or Bloch-type equation. The general solution of eqn (14) can be derived in many different ways. One is the probabilistic representation of solutions by virtue of the well-known Feinman–Kac formula by using path-integral methods.[2,3,11,12] The other is an eigenmode expansion.[13,14] In first case, the solutions thus obtained are, unfortunately, formal and are very difficult for engineering applications. In the second case, the differential operator in the FPK equation is not, in general, self-adjoint, and makes the formalism of variational schemes (such as Rayleigh–Ritz) more complicated: eigenvalues are not real, nor are the right and left eigenfunctions equal. In Ref.[14], the schemes of two types were considered. In the first type, a Hermitian operator is constructed from the (non-Hermitian) FPK operator, and one then proceeds as a variational calculation in quantum mechanics. The second type is based on a soluble approximation of the FPK operator and the pertubation theory based there on. This situation gives rise to many simulating opportunities for the development of approximate procedures to analyze such equations. Several numerical techniques exist in the literature for the analysis of nonlinear problems. Widley used procedures on basis-set (cumulant) expansion, path-integral techniques, iterative time-dependent propagation schemes, stochastic computer simulation, and moment expansion. Each of them has its own advantages when applied to the nonlinear FPK equation. This problem is discussed in detail in Refs[3–5,10,15].

Remark 2. The focus of this remark is on one of the most debated problems in the field of stochastic dynamic system theory: the derivation of the best FPK equation for colored-noise-driven stochastic dynamic system.[3–5,11,12,32–39] The problem is to derive an FPK-type evolution equation for the probability density function of the process described by means of substituting a linear combination of uncorrelated colored noise for correlated colored noise:

$$\dot{x}=G(x)+g^c(x)p(t)+g^w(x)q(t);\ \dot{p}=-\frac{1}{\tau}p+\frac{1}{\tau}q_1(t) \qquad (15)$$

where $x=(x_1,x_2,\ldots,x_n)$ and $G(x)$, $g(x)$ are nonlinear functions; the superscripts 'c' and 'w' denote, respectively, the functional coefficients of the colored and white fluctuations. In this case the fluctuations $q(t)$ are δ-correlated in time, $\langle q_i(t)q_j(t')\rangle=2D_w\delta_{ij}\delta(t-t')$ with intensity D_w. The fluctuations $p(t)$ are colored and are assumed to have an exponential correlation function, $\langle p_i(t)p_j(t')\rangle=(D/\tau)\delta_{ij}\exp\{-|t-t'|/\tau\}\rightarrow 2D_w\delta_{ij}\delta(t-t')$ as $\tau\rightarrow 0$, where D is the noise strength and τ is the noise correlation time. If the driving noise is Gaussian and δ-function correlated, the process in eqn (15) is Markovian and described by FPK equation (an equation for the probability density of x which is local in time and space). If the noise has finite correlation time, process x becomes a non-Markovian and its evolution equation is in general nonlocal both in time and space. Recently, Fulinskí and Telejko[32] have investigate the system driven by colored additive and multiplicative white noises and have shown that the presence of correlation between the noises changes the dynamics of the system. The stochastic dynamics of a system such as eqn (15) driven by δ-correlated ($\langle q(t)q_1(t')\rangle=2\lambda\sqrt{D\alpha}\delta(t-t')$, $\langle q(t)q(t')\rangle=2\alpha\delta(t-t')$, $\langle q_1(t)q_1(t')\rangle=2D\delta(t-t')$) and colored ($\langle q(t)p(t')\rangle=\lambda\sqrt{(QD)}/\tau\ \exp(-|t-t'|/\tau)$, $\langle q(t)q(t')\rangle=D/\tau\ \exp(-|t-t'|/\tau)$, $g^w=1$, $\langle p(t)p(t')\rangle=Q/\tau\exp(-|t-t'|/\tau)$) noise in Refs[35,36] was studied. The simultaneous consideration of additive and multiplicative correlated white noises can induce[36] a very large suppression (or with an anticorrelation the opposite effect of a large enhancement) of the forward transition rate in bistable systems.

(2) The *stochastic mechanics* formalism[40,41] can be illustrated as follows in the simple case of a classical dynamic system[42] defined by the Lagrangian:

$$L(x,\dot{x})=\frac{1}{2}m\dot{x}_i\dot{x}_i-U(x) \qquad (16)$$

with $U(x)$ time-independent. An exact ground-state wave function $\psi_0(x, t)$ can be parametrized as:

$$\psi_0(x,t) = \exp\left\{\frac{[iE_0t + \omega(x)]}{\hbar}\right\} \tag{17}$$

with $\omega(x)$ a real function and satisfying the Riccati equation

$$\frac{1}{2m}\left(\frac{\partial}{\partial x_i}\omega(x)\right)^2 - \frac{\hbar}{2m}\frac{\partial^2}{\partial x_i^2}\omega(x) + E_0 = U(x). \tag{18}$$

Quantum dynamics formulated at imaginary time $s = it$ and according to eqn (18), the Schödinger equation becomes:

$$\frac{\partial}{\partial s}\psi(x,s) = \frac{\hbar}{2m}\frac{\partial^2}{\partial x_i^2}\psi(x,s) - \frac{1}{\hbar}\left[\frac{1}{2m}\left(\frac{\partial}{\partial x_i}\omega(x)\right)^2 - \left(\frac{\hbar}{2m}\frac{\partial^2}{\partial x_i^2}\omega(x)\right) + E_0\right]\psi(x,s) \tag{19}$$

Setting now:

$$\psi(x,t) \equiv p(x,s)\exp\left\{-\frac{[E_0s - \omega(x)]}{\hbar}\right\} \tag{20}$$

it follows that eqn (19) takes the form

$$\frac{\partial}{\partial s}p(x,s) = \frac{\hbar}{2m}\frac{\partial^2}{\partial x_i^2}p(x,s) - \frac{\partial}{\partial x_i}\left[-\frac{1}{2m}\left(\frac{\partial}{\partial x_i}\omega(x)\right)p(x,s)\right] \tag{21}$$

which is just an FPK equation. The function $p(x,s)$ can be regarded as the probability density of a classical stochastic diffusion process with diffusion constant $\hbar/2m$ and drift velocity $(-\nabla\omega(x)/m)$. In this case, quantum dynamics at imaginary time is equivalent to a certain classical diffusion process with conserved particle number. This process can alternatively be described by Langevin equation:

$$\frac{\partial}{\partial s}\xi_i(s) = .-\frac{1}{2m}\frac{\partial}{\partial x_i}\omega(x)\bigg|_{x=\xi(s)} + \left(\frac{\hbar}{m}\right)^{1/2}\eta_i(s) \tag{22}$$

where $\eta(s) = \{\eta_i(s)\}|_{1\le i\le N}$ is a Gaussian white noise defined by the probability measure $\mu[\eta(s)] = \eta(s)\exp(-1/2\int_{-\infty}^{+\infty} \mathrm{d}s\, \eta_i(s)\eta_i(s))$.The generalized Langevin quantization of a classical dynamic system defined by dissipative Lagrangian:

$$L(x,\dot{x},t) = \frac{1}{2}m\dot{x}_i\dot{x}_i + \Omega_i(x,t)\dot{x}_i - U(x,t) \tag{23}$$

rests upon the Langevin equation

$$\frac{\partial}{\partial s}\xi_i(s) = -\frac{1}{m}\left(\frac{\partial}{\partial x_i}S(x,s) + i\Omega(x)\right)\bigg|_{x=\xi(s)} + \left(\frac{\hbar}{m}\right)^{1/2}\eta_i(s) \tag{24}$$

controlled by $S(x, s)$, where $S(x, s)$ denotes an arbitrary integral of the imaginary-time Hamilton–Jacobi equation associated with Lagrangian eqn (23):

$$\frac{\partial}{\partial s}S(x,s) + \frac{1}{2m}\left(\frac{\partial}{\partial x_i}S(x,s) + i\Omega(x,s)\right)^2 - U(x,t) = 0 \tag{25}$$

Accordingly, the imaginary-time Schrödinger equation corresponding to the Lagrangian eqn (23) is real and can be written as:[42]

$$\frac{\partial}{\partial s}\psi(x) = \left(\frac{\hbar}{2m}\frac{\partial^2}{\partial x_i^2} - \frac{\partial}{\partial x_i}i\Omega(x,s)\right)\psi(x,s) + \frac{1}{\hbar}\left[\frac{\hbar}{2m}\left(\frac{\partial}{\partial x_i}i\Omega(x,s)\right) - \frac{1}{2m}\Omega_i(x,s)^2 - U(x,t)\right] \times \psi(x,s) = 0 \tag{26}$$

Manifestly, eqn (26) can be viewed as the FPK equation for a classical stochastic diffusion process with diffusion constant $\hbar/2m$ and drift velocity $(-\nabla i\Omega(x, s)/m)$. Therefore, the FPK description of the diffusion process in question yields the usual (Schrödinger) formulation of quantum dynamics as $s = it$.

Suppose $\{\psi(x, t); 0 \le t \le T\}$ is a never-vanishing solution of the Schrödinger equation:

$$\frac{\partial\psi}{\partial t} = \frac{i\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} - \frac{i}{\hbar}V(x)\psi \tag{27}$$

then $S_q(x, t) = \hbar/i \log \psi(x, t)$ satisfies

$$\frac{\partial S_q}{\partial t} + H = 0;\ H = \frac{1}{2m}\nabla S_q\nabla S_q + V(x) - \frac{i\hbar}{2m}\Delta S_q \tag{28}$$

This is the Hamilton–Jacobi equation of stochastic mechanics. If $p(x) = |\psi(x, T)|^2$, then there exist Markov diffusion process $\{q(t); 0 \le t \le T\}$ with the quantum drift given by $v_q(q(t), t) = 1/m\nabla S_q(q(t), t)$. Letting:

$$\psi(x,t) = \exp\left[R(x,t) + \frac{i}{\hbar}S(x,t)\right] \tag{29}$$

we see[43] that process q has the current velocity $v_q(q(t), t) = 1/m\ \nabla S_q(q(t),t)$ and the osmotic velocity $u(q(t), t) = (\hbar/m)\nabla R(q(t), t)$. It is, namely, the Nelson process associated to the solution $\{\psi(x, t);\ 0 \le t \le T\}$ of the Schrödinger equation. The Nelson process $q(t)$ and the momentum process $p(t) = \nabla S_q((x\ (t),t)$, associated to a particular solution of the Hamilton–Jacobi eqn (28) satisfy[40,43,44] the equations:

$$\mathrm{d}q(t) = \nabla_y H(q(t), p(t))\,\mathrm{d}t + \mathrm{d}w,$$
$$\mathrm{d}p(t) = -\nabla_x H(q(t), p(t)) + \mathbf{D}(q(t), t)\,\mathrm{d}w \tag{30}$$

where the 3×3 matrix $\mathbf{D}$ has as its (ij) entry $d_{ij}(q(t),t) = ((\partial^2)/(\partial x_i\partial x_j)S_q(q(t), t)$. Eqn (30) has the form of Hamilton's canonical equations with a random perturbation.[29,30,40,43,44]

For the following generalized stochastic Hamilton–Jacobi equation:[40,45]

$$\mathrm{d}_tS(x) = \left[c(x,t) + \frac{1}{2}|\nabla S_q(x)|^2\right]\mathrm{d}t + \sum_{j=1}^{m} k^j(t,x)\,\mathrm{d}w_t^j \tag{31}$$

where w_t^j is an m-dimensional Brownian motion on accordingly probability space and we can write a stochastic

Schrödinger equation corresponding to eqn (31) as:[45]

$$i\hbar\, d_t\psi_t(x) = \left[-\frac{1}{2}\hbar^2\Delta\psi_t + c(x,t)\psi_t(x)\right] dt + \psi_t \sum_{j=1}^{m} k^j(t,x)\, \partial w_t^j \quad (32)$$

where ∂w_t^j is the Stratonovich stochastic differential.

The structures of stochastic Hamiltonian mechanics in Refs.[40,41,44] was studied in detail. Generalized stochastic relativistic mechanics in Refs[46–50] was discussed. In Part 4, a model of stochastic relativistic Hamilton–Jacobi equation as background for quantum and relativistic diffusion processes is considered.

The models of the stochastic Schrödinger eqn (32) was developed in Refs [45,51–53] for optimal control processes with nondemolition measurement. In Part 5, we discuss the structure of these models and interrelations with the stochastic Hamilton–Jacobi eqn (31) and mathematical models of classical and quantum flows.[54–56] Interrelations of the nonlinear Schrödinger equations and FPK equations with stochastic optimal control in Refs[57–59] is discussed (see Section A.4). Application of this approach in Ref.[60] is demonstrated.

3 ASYMPTOTIC METHOD IN STATISTICAL ANALYSIS OF NONLINEAR SYSTEMS ON THE BASIS OF THE FPK EQUATION

As an example of statistical analysis on the basis of the FPK equation, we consider a nonlinear dynamic system described as:

$$\ddot{x} + [2\beta_0 + 2\mu\alpha(t)]\dot{x} + \Omega^2[1 + 2\sigma\chi(t)] + \varepsilon F(x,\dot{x},\ddot{x}) = \eta(t) \quad (33)$$

where $\alpha(t)$, $x(t)$, and $\eta(t)$ are stationary stochastic correlated processes with zero mean values and bounded power spectral densities $S_\alpha(\omega)$, $S_\chi(\omega)$ and $S_\eta(\omega)$ and autocorrelation functions $R_\alpha(\tau)$, $R_\chi(\tau)$ and $R_\eta(\tau)$, respectively; the nonlinear function $F(x,\dot{x},\ddot{x})$ has an arbitrary form; ε is a parameter with a small value; β_0, μ, Ω^2, σ and ν are constants with arbitrary real values. Eqn (33) describes many nonlinear systems subjected to different random excitations such as earthquakes, vibration loads, random friction forces and so on[16–28].

Remark 3. In eqn (33), we assume that random parameters $\alpha(t)$, $\chi(t)$ and $\eta(t)$ do not lead to large changes in amplitude and phase of output during the time period of motion. We also assume that the parameter β_0 for the friction force is small and eqn (33) describes a narrow-band dynamic system. Under these assumptions, the output of the system in eqn (33) is likely to be a quasi-harmonic vibration with a slow change in amplitude and phase with time. The numerical simulation of such a system subjected to a real strong nonstationary random excitation as an earthquake accelerogram confirms this state.[16,17]

Consistent with to the asymptotic method of Bogolyubov–Mitropolsky,[1,16] we assume:

$$x(t) = A(t)\cos(\Omega t + \psi(t)) \quad (34)$$

and

$$\dot{x}(t) = -\Omega A(t)\sin(\Omega t + \psi(t)) \quad (35)$$

where A and ϕ are amplitude and phase, respectively.

The equations in a 'standard form' [1,16] for the system [eqn (33)] are:

$$\begin{cases} \dot{A}(t) = \dfrac{\dot{x}(t)}{\Omega^2 A(t)}[-2\beta_0\dot{x}(t) - 2\mu\alpha(t)\dot{x}(t) - 2\sigma\Omega^2\chi(t)x(t) \\ \quad + \varepsilon F(x,\dot{x},\ddot{x}) + \nu\eta(t)] \\ \dot{\psi}(t) = -\dfrac{x(t)}{\Omega A^2(t)}[-2\beta_0\dot{x}(t) \\ \quad - 2\mu\alpha(t)\dot{x}(t) - 2\sigma\Omega^2\chi(t)x(t) + \varepsilon F(x,\dot{x},\ddot{x}) + \nu\eta(t)] \end{cases} \quad (36)$$

The vibrational functions contained in the regular and fluctuational terms of eqn (36) can be extracted with any degree of accuracy. In the first approach, vibrational functions are extracted with simple averaging within the time period.

It is possible to define a nonvibrational equation with higher degree of accuracy using a special variation of the asymptotic method suggested in Ref.[1]. In this method, the two parts of eqn (36) are rewritten in the following form:[16]

$$\dot{A} = \beta_0 G^* + G;\ \dot{\psi} = \beta_0 H^* + H, \quad (37)$$

where G^* and H^* are regular terms, and G and H are fluctuational terms.

Taking off the fluctuational terms:

$$\dot{A} = \beta_0 G^*;\ \dot{\psi} = \beta_0 H^* \quad (38)$$

and expanding functions G^* and H^* in series on a small parameter β_0, we obtain

$$\dot{A} = \beta_0 G_1 + \beta_0^2 G_2 + \ldots;\ \dot{\psi} = \beta_0 H_1 + \beta_0^2 H_2 + \ldots \quad (39)$$

The functions G_1 and H_1 are evaluated by simply averaging the functions G^* and H^* on the parameter $\Phi = \Omega t + \psi(t)$ as:

$$G_1 = \langle G^*\rangle_\Phi;\ H_1 = \langle H^*\rangle_\Phi \quad (40)$$

The functions G_2 and H_2 determine nonvibrational terms from an expression as:

$$\left(\frac{\partial G^*}{\partial A}u_1 + \frac{\partial G^*}{\partial \psi}v_1\right);\ \left(\frac{\partial H^*}{\partial A}u_1 + \frac{\partial H^*}{\partial \psi}v_1\right) \quad (41)$$

and the functions u_1 and v_1 are, respectively, determined from the expressions

$$\frac{\partial u_1}{\partial \Phi} = \frac{1}{\Omega}[G^* - \langle G^*\rangle_\Phi];\ \frac{\partial v_1}{\partial \Phi} = \frac{1}{\Omega}[H^* - \langle H^*\rangle_\Phi] \quad (42)$$

To extract the vibrational functions from the fluctuational terms, we use the modified method suggested in Ref.[16].

Present the fluctuational terms in eqn (36) as the sum of means (m_1, m_2) and centered random components ($\xi_1(t)$,

$\xi_2(t)$) with δ-similar autocorrelation function:

$$G(t) = \frac{\dot{x}(t)}{\Omega^2 A(t)}\nu\eta(t) - 2\sigma A^{-1}(t)\chi(t)x(t)\dot{x}(t) - 2\mu\alpha(t)\dot{x}^2(t)A^{-1} = \nu\Omega^{-1}\eta(t)\sin\Phi + \sigma\Omega A\sin 2\Phi\chi(t) - \mu\alpha(t)A(1-\cos 2\Phi) = m_1 + \xi_1(t) \quad (43)$$

and

$$H(t) = -\frac{x(t)}{\Omega A^2(t)}\nu\eta(t) + 2\sigma\Omega A^{-2}(t)\chi(t)x^2(t) + 2\mu\alpha(t)\dot{x}^2(t)x(t)\Omega^{-1}A^{-2} = -\nu\Omega^{-1}A^{-1}\eta(t)\cos\Phi + \sigma\Omega(1+\cos 2\Phi)\chi(t) - 2\mu\alpha(t)\sin 2\Phi = m_2 + \xi_2(t) \quad (44)$$

where

$$m_1 = \left\langle -\frac{\nu}{\Omega}\eta(t)\sin\Phi + \sigma\Omega A\chi(t)\sin 2\Phi - \mu\alpha(t)A(1-\cos 2\Phi)\right\rangle \quad (45)$$

$$m_2 = \left\langle -\frac{\nu}{\Omega A(t)}\eta(t)\cos\Phi + \sigma\Omega\chi(t)(1+\cos 2\Phi) - \mu\alpha(t)\sin 2\Phi\right\rangle \quad (46)$$

Intensity coefficients of $\xi_1(t)$ and $\xi_2(t)$ processes are:

$$K_1 = \int_{-\infty}^{\infty}\langle\xi_1(t)\xi_1(t+\tau)\rangle\,d\tau = \int_{-\infty}^{\infty}K[G(t),G(t+\tau)]\,d\tau + \int_{-\infty}^{0}K[G(t),H(t+\tau)]\,d\tau + \int_{-\infty}^{0}K[G(t+\tau),H(t)]\,d\tau = \frac{\nu^2}{2\Omega^2}S_\eta(\Omega) + \frac{\sigma^2\Omega^2A^2}{2}S_\chi(2\Omega) + \mu^2A^2[S_\alpha(0)+S_\alpha(2\Omega)] \quad (47)$$

and

$$K_2 = \int_{-\infty}^{\infty}\langle\xi_2(t)\xi_2(t+\tau)\rangle\,d\tau = \int_{-\infty}^{\infty}K[H(t),H(t+\tau)]\,d\tau + \int_{-\infty}^{0}K[H(t),G(t+\tau)]\,d\tau + \int_{-\infty}^{0}K[H(t+\tau),G(t)]\,d\tau = \frac{\nu^2}{2\Omega^2 A}S_\eta(\Omega) + \sigma^2\Omega^2\left[S_\chi(0) + \frac{1}{2}S_\chi(2\Omega)\right] + \frac{1}{2}\mu^2 S_\alpha(2\Omega) \quad (48)$$

where $K[\dots]$ is an autocorrelation function; $S_\alpha(\dots)$, $S_\chi(\dots)$ and $S_\eta(\dots)$ are power spectral density functions of $\alpha(t)$, $\chi(t)$ and $\eta(t)$ processes on corresponding frequencies.

The expressions $\int_{-\infty}^{+\infty}K[G(t+\tau),H(t)]\,d\tau$ and $\int_{-\infty}^{0}K[G(t),H(t+\tau)]\,d\tau$ give solely vibrational terms. In the first approach, therefore, the random processes $\xi_1(t)$ and $\xi_2(t)$ may be considered as statistically independent stochastic processes. The means m_1 and m_2 can be calculated by the Stratonovich formula[1] as:

$$m_1 = \int_{-\infty}^{0}K\left[\frac{\partial G(t)}{\partial A}, G(t+\tau)\right]d\tau + \int_{-\infty}^{0}K\left[\frac{\partial G(t)}{\partial\psi}, H(t+\tau)\right]d\tau = \frac{\nu^2}{4\Omega^2 A}S_\eta(\Omega) + \frac{3\sigma^2\Omega^2 A}{4}S_\chi(2\Omega) + \mu^2 A\left[S_\alpha(0) + \frac{3}{2}S_\alpha(2\Omega)\right] \quad (49)$$

and

$$m_2 = \int_{-\infty}^{0}K\left[\frac{\partial H(t)}{\partial A}, G(t+\tau)\right]d\tau + \int_{-\infty}^{0}K\left[\frac{\partial H(t)}{\partial\psi}, H(t+\tau)\right]d\tau = \sigma^2\Omega^2\int_0^\infty R_\chi(\tau)\sin 2\Omega\tau\,d\tau + 2\mu^2\int_0^\infty R_\alpha(\tau)\sin 2\Omega\tau\,d\tau \quad (50)$$

Note the following two special cases:

(1) Processes $\alpha(t)$, $\chi(t)$ and $\eta(t)$ are white noises:

$$S_\alpha(2\Omega) = S_\chi(2\Omega) = S_\eta(\Omega);$$

$$m_2 = 0;\; m_1 = \frac{\nu^2}{4\Omega^2 A} + \frac{3\sigma^2\Omega^2 A}{4} + \frac{5}{2}\mu^2 A;$$

$$K_1 = \frac{\nu^2}{2\Omega^2} + \frac{\sigma^2\Omega^2A^2}{2} + 2\mu^2A^2;$$

$$K_2 = \frac{\nu^2}{2\Omega^2 A} + \frac{3}{2}\sigma^2\Omega^2 + \frac{\mu^2}{2} \quad (51)$$

(2) For $\mu = 0$, we obtain the result as in Ref.[16].

Calculating G_1, G_2, H_1, H_2, m_1, m_2, K_1 and K_2, the amplitude A and phase ψ contained in G and H become fixed (nonrandom).[1,16] Thus, the equations for the amplitude and phase with extracted vibrational terms in the second approach can be written as:

$$\dot{A} = \beta_0 G_1 + \beta_0^2 G_2 + m_1 + \xi_1(t) \quad (52)$$

and

$$\dot{\psi} = \beta_0 H_1 + \beta_0^2 H_2 + m_2 + \xi_2(t) \quad (53)$$

Remark 4. If the correlation time τ_c for $\alpha(t)$, $\chi(t)$ and $\eta(t)$ are, to a marked degree, less than relaxation time τ_r, for the

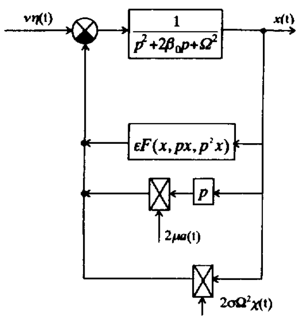

Fig. 1. Block diagram of nonlinear parametric system.

output amplitude and phase of the dynamic system of eqn (33), while the responsed time value of the system, to a marked degree, surpasses $-\tau_c$, then it is possible to apply workable stochastic methods by replacing real excitation processes $\alpha(t)$, $\chi(t)$ and $\eta(t)$ by equivalent δ-correlation processes. In this case, we can use the Markov stochastic process approach and the FPK equations for the definition of probability density functions of the output amplitude and the phase of the system represented by eqn (33). For the system in eqns (52) and (53), the FPK equation for the definition of a two-dimensional mutual probability density function of the amplitude and phase (A, ψ, t) is:

$$\begin{aligned}\frac{\partial p(A,\psi,t)}{\partial t} = &-\frac{\partial}{\partial A}\{[\beta_0 G_1 + \beta_0^2 G_2 + m_1]p(A,\psi,t)\}\\ &-\frac{\partial}{\partial\psi}\{[\beta_0 H_1 + \beta_0^2 H_2 + m_2]p(A,\psi,t)\}\\ &+\frac{\partial^2}{\partial A_2}[K_1 p(A,\psi,t)] + \frac{\partial^2}{\partial\psi^2}[K_2 p(A,\psi,t)]\end{aligned} \qquad (54)$$

The preference of this method is in the independence of eqn (52) from eqn (53) for any type of nonlinearities of eqn (33). In this case, we investigate statistical characteristics of amplitude $A(t)$ separately on basis of the one-variable FPK equation for $p(A, t)$ as:

$$\begin{aligned}\frac{\partial p(A,t)}{\partial t} = &-\frac{\partial}{\partial A}\{[\beta_0 G_1 + \beta_0^2 G_2 + m_1]p(A,t)\}\\ &+\frac{1}{2}\frac{\partial^2}{\partial A^2}[K_1 p(A,t)]\end{aligned} \qquad (55)$$

The solution of the one-variable FPK equation is simpler than that of the two-variable the FPK equation. Application of the variational Bubnov–Galerkin method for solving eqn (55) is shown later in the concrete examples.

Example 1. Consider the dynamic system of eqn (33) as:

$$\begin{aligned}&\ddot{x} + [2\beta_0 + 2\mu\alpha(t)]\dot{x} + \Omega_0^2[1 + 2\sigma\chi(t)]x + \varepsilon F(x,\dot{x},\ddot{x})\\ &= \nu\eta(t)\end{aligned} \qquad (56)$$

where $\varepsilon F(x,\ \dot{x},\ \ddot{x}) = 2k(\ddot{x}x + \dot{x}^2)x + \gamma_1 x^3 + \gamma_2 x^5 + \varepsilon\dot{x}x^2$; ε, κ, γ_1 and γ_2 are constants. Fig. 1 show a general block diagram of dynamic system eqn (56).

The nonlinear dynamic system that is eqn (56) describes the behavior of different systems under random excitations, such as a structure under vertical and horizontal accelerograms of a strong earthquake, an automatic control system with stochastic feedback gains, a stall of tracking in a nonlinear automatic control system, and so on. The investigation of statistical characteristics of eqn (56) has independent importance in this case.

Using eqns (34)–(44), the amplitude and a phase for eqn (56) are obtained:

$$\begin{aligned}\dot{A}(t) = \frac{1}{\Omega^2 A}\Bigg[&-\beta_0\frac{\Omega^2 A^2}{2}(1-\cos 2\Phi) - \frac{k\Omega^3 A^4}{2}\\ &\times\sin 4\Phi + \frac{\gamma_1\Omega A^4}{4}\left(\sin 2\Phi + \frac{1}{2}\sin 4\Phi\right)\\ &+\frac{\gamma_2\Omega A^6}{16}\left(\frac{1}{2}\sin 6\Phi + 2\sin 4\Phi + \frac{5}{2}\sin 2\Phi\right)\\ &-\frac{\varepsilon\Omega^2 A^4}{4}(1-\cos 4\Phi)\Bigg] + \sigma A\Omega\chi(t)\sin 2\Phi\\ &-\frac{\nu}{\Omega}\eta(t)\sin\Phi - \mu\alpha(t)A(1-\cos 2\Phi)\end{aligned} \qquad (57)$$

and

$$\begin{aligned}\dot{\psi}(t) = -\frac{1}{\Omega A^2}\Bigg[&\frac{1}{2}\beta_0\Omega A^2\sin 2\Phi + 2k\Omega^2 A^4\\ &\times\left(\frac{1}{2} + \cos 2\Phi + \frac{1}{2}\cos 4\Phi\right)\\ &-\frac{\gamma_1 A^4}{8}(3 + 4\cos 2\Phi + \cos 4\Phi)\\ &-\frac{\gamma_2 A^6}{16}\left(\frac{1}{2}\cos 6\Phi + 3\cos 4\Phi + \frac{15}{2}\cos 2\Phi + 5\right)\\ &+\varepsilon\Omega A^4\left(\frac{1}{8}\sin 4\Phi + \frac{1}{4}\sin 2\Phi\right)\\ &+\frac{\Omega^2 A^2}{2}(1+\cos 2\Phi)\Bigg] + \sigma\Omega\chi(t)(1+\cos 2\Phi)\\ &-\frac{\nu\eta(t)}{\Omega A}\cos\Phi - \mu\alpha(t)\sin 2\Phi\end{aligned} \qquad (58)$$

In eqns (57) and (58), the vibrational functions are extracted with transition from simple averaging over a time period to the second approach shown in eqns (40) and (42).

Remark 5. As mentioned above, averaging over a time period due to the availability of a fluctuational excitation is possible, due to the assumption that the correlation time τ_c of $\psi(t)$ is less than the period of normal mode of vibration. Tacking off the vibrational functions from eqns (57) and

(58), the basis of simple averaging leads to the disappearance of all the nonlinear factors. This explains why it is necessary to transit from simply averaging over a time period to the second approach. The vibrational functions can be extracted separately from the fluctuational and regular terms. For all these it is necessary to take into account correlation connection between an excitation and a phase.

The vibrational functions extracted from the regular terms in the second approach as:

$$\begin{cases} G^* = -a_1 A - I_1 A^3 + I_2 A^5 + I_3 A^7, \\ H^* = -a_4 - a_2 A^2 + a_3 A^4 + a_5 A^6 - a_6 A^6 \end{cases} \tag{59}$$

where

$$a_1 = \beta_0;\ a_2 = \frac{k\Omega}{2} - \frac{3\gamma_1}{8\Omega} + \frac{\varepsilon\beta_0}{4\Omega};\ a_3 = \frac{5\gamma_2}{16\Omega} - \frac{11k^2\Omega}{16} + \frac{23k\gamma_1}{32\omega} - \frac{51\gamma_1^2}{256\Omega^3} - \frac{11\varepsilon^2}{256\Omega};\ a_4 = \frac{\beta_0^2}{2\Omega};\ a_5 = \frac{45\gamma_2 k}{64\Omega} - \frac{55\gamma_1\gamma_2}{128\Omega^3};\ a_6 = \frac{1325\gamma_2}{6144\Omega^3};\ I_1 = \frac{\varepsilon}{8} - \frac{\beta_0 k}{2};$$

$$I_2 = \frac{\varepsilon\gamma_1}{32\Omega^2};\ I_3 = \frac{5\varepsilon\gamma_1}{256\Omega^2} \tag{59a}$$

Thus 'the shortened equations' eqns (52) and (53) with extracted vibrational functions can be written as:

$$\dot{A} = -a_1 A - I_1 A^3 + I_2 A^5 + I_3 A^7 + m_1 + \xi_1(t) \tag{60}$$

and

$$\dot{\psi} = -a_4 - a_2 A^2 + a_3 A^4 + a_5 A^6 + m_2 + \xi_2(t) \tag{61}$$

The functions m_1, m_2, $\xi_1(t)$ and $\xi_2(t)$ are evaluated using eqns (45)–(50). Eqns (60) and (61) can be used to investigate different special cases with practical importance.

The FPK equation for eqn (60) can be obtained using eqn (55) as:

$$\frac{\partial p(A,t)}{\partial t} = -\frac{\partial}{\partial A}\left[\frac{a^7}{2}A^{-1} - \left(a_1 - \frac{3a_8}{2} - a_9 - a_{10}\right)A - I_1 A^3 + I_2 A^5 + I_3 A^7\right]p(A,t) + \frac{1}{2}\frac{\partial^2}{\partial A^2}[(a_7 + a_8^* A^2)p(A,t)] \tag{62}$$

where

$$a_7 = \frac{\nu^2 S_\eta(\Omega)}{2\Omega^2};\ a_8 = \frac{\sigma^2\Omega^2 S_\xi(2\Omega)}{2};$$

$$a_9 = \mu^2 S_a(0);\ a_{10} = 1.5\mu^2 S_a(2\Omega);$$

$$a_8^* = a_8 + a_9 + \frac{2}{3}a_{10}$$

The stationary distribution $p_{st}(A, t)$ as a solution of eqn (62) can be obtained as:

$$p_{st}(A) = \frac{C}{(a_7 + a_8^* A_2)}\exp \times \left[2\int \frac{[0.5a_7 A^{-1} - (a_1 - 1.5a_8 - a_9 - a_{10})A - I_1 A^3 + I_2 A^5 + I_3 A^7]}{(a_7 + a_8^* A^2)}\,\mathrm{d}A\right] \tag{63}$$

The constant C is evaluated from a natural condition of a normalization as:

$$\int_{-\infty}^{\infty} p_{st}(A)\,\mathrm{d}A = 1 \tag{64}$$

To obtain a nonstationary distribution $p(A,t)$ as a solution of eqn (62), we use the Bubnov–Galerkin method. This approach is described in Ref.[16]. Other approaches to solving the FPK equations from the point of view of Lie group analysis, generalized symmetries of partial differential equations and connection with the Schrödinger equation are described in Appendix A.

4 THE METHOD OF A NONSTATIONARY SOLUTION OF FPK EQUATION

The nonstationary solution of eqn (62) is defined as:

$$p(A,t) = p_{st}(A) + \sum_{m=1}^{n} T_m(t) p_m(A) \tag{65}$$

where $p_m(A)$ is an approximating function with boundary conditions $p_m(0, t) = 0$; $p_m(\infty, t) \to 0$.

Assuming function $p_m(A)$ is given as:

$$p_m(A) = \frac{A}{\sigma_1^2}\exp\left(-\frac{A^2}{2\sigma_1^2}\right)L_m\left(\frac{A^2}{2\sigma_1^2}\right);$$

$$\sigma_1^2 = \frac{0.5a_7}{a_1 - 1.5a_8 - a_9 - a_{10}}$$

$L_m(\ldots)$ is Laguerre's polynomials that corresponds to solution of linear forced vibration. The function $T_m(t)$ is evaluated by a series of equations as:

$$\sum_{m=1}^{n} [\alpha_{mj}\dot{T}_m(t) + \beta_{mj}T_m(t)] = 0\ (j = 1, 2, \ldots, n) \tag{67}$$

where

$$\alpha_{mj} = \frac{\sqrt{2}}{\sigma_1}\int_0^\infty x^{1/2}\mathrm{e}^{-2x}L_m(x)L_j(x)\,\mathrm{d}x;$$

$$\beta_{mj} = -\frac{a_7}{2\sigma_1^3}N_{1mj} - \frac{2\sqrt{2}}{\sigma_1}a_8^* N_{5mj}$$

Table 1. Coefficent values $\alpha_{mj}\sigma, N_{1mj}, N_{2mj}, N_{3mj}, N_{4mj}, N_{5mj}$ for j = 1, 2, 3

m	$\alpha_{mj}\sigma$			N_{1mj}			N_{2mj}		
1	0.912	0.116	0.0297	− 0.788	− 1.858	− 4.858	0.0763	0.319	0.285
2	0.116	0.137	0.1	− 0.0336	− 0.524	0.99	− 0.254	− 0.283	− 0.082
3	0.0297	0.1	0.109	− 1.002	− 9.062	− 65.33	− 0.41	− 0.206	35.5
m	N_{3mj}			N_{4mj}			N_{5mj}		
1	0.717	1.357	0.815	5.77	5.77	− 26.6	0.51	2.34	1.92
2	0.599	1.695	1.3	7.59	3.39	5.32	2.16	2.40	4.61
3	− 0.424	− 0.113	1.865	− 4.42	− 3.51	22.9	2.72	4.00	11.6

$$- (a_1 - 1.5a_8 - a_9 - a_{10})\sigma_1^{-1}N_{2mj}$$
$$+ I_1\sigma_1 N_{3mj} - I_2\sigma_1^3 N_{4mj};$$

$$N_{1mj} = 2\sqrt{2}\int_0^\infty \frac{d^2}{d^2x}[x^{1/2}e^{-x}L_m(x)]xe^{-x}L_j(x)\,dx$$
$$-\sqrt{2}\int_0^\infty \frac{d}{dx}[e^{-x}L_m(x)]x^{1/2}e^{-x}L_j(x)\,dx;$$
$$N_{2mj} = 2\sqrt{2}\int_0^\infty \frac{d}{dx}[xe^{-x}L_m(x)]x^{1/2}e^{-x}L_j(x)\,dx;$$
$$N_{3mj} = 4\sqrt{2}\int_0^\infty \frac{d}{dx}[x^3e^{-x}L_m(x)]x^{1/2}e^{-x}L_j(x)\,dx;$$
$$N_{4mj} = 8\sqrt{2}\int_0^\infty \frac{d}{dx}[x^5e^{-x}L_m(x)]x^{1/2}e^{-x}L_j(x)\,dx;$$
$$N_{5mj} = \int_0^\infty \frac{d^2}{dx^2}[x^{3/2}e^{-x}L_m(x)]xe^{-x}L_j(x)\,dx$$

The values of coefficients $N_{1mj},\ldots,N_{5mj}$ for j = 1, 2, 3 obtained in Refs[16,17] are shown in Table 1.

The solution of eqn (67) is $T_m(t) = C\exp\{\lambda_m t\}$. The values of $\lambda_1, \lambda_2, \ldots, \lambda_m$ are determined from the following equation:

$$\begin{vmatrix} \alpha_{11}\lambda + \beta_{11}, & \cdots & \alpha_{n1}\lambda + \beta_{n1} \\ \vdots & \vdots & \vdots \\ a_{1n}\lambda + \beta_{1n}, & \cdots & a_{nn}\lambda + \beta_{nn} \end{vmatrix} = 0 \qquad (68)$$

and the values of $C_1^{(k)}, C_2^{(k)}, \ldots, C_m^{(k)}$, corresponding to λ_k are obtained from:

$$\sum_{m=1}^{n} [\alpha_{mj}\lambda_k(t) + \beta_{mj}]C_m^{(k)} = 0 \; (j = 1, 2, \ldots, n) \qquad (68a)$$

Remark 6. Coefficients $C_m^{(k)}$ are evaluated to any arbitrary factor D_k that is $C_1^{(k)} = D_k K_1^{(k)}$, $C_2^{(k)} = D_k K_2^{(k)}$, ..., $C_m^{(k)} = D_k K_m^{(k)}$ where $K_1^{(k)}, K_2^{(k)}, \ldots, K_m^{(k)}$ are the minors of elements from first row of the determinant in eqn (68).

The complete solution of eqn (67) define as:

$$T_m(t) = \sum_{k=1}^{n} D_k K_m^{(k)} e^{\lambda_k t} \qquad (69)$$

The coefficients D_k are obtained from an initial states at $t = 0$, $p_m(A, 0) = p_0(A)$.

Substituting eqn (69) into eqn (65), the nonstationary solution at t = 0 is obtained as:

$$p(A,0) = p_0(A) = p_{st}(A) + \sum_{m=1}^{n}\sum_{k=1}^{n} D_k K_m^{(k)} p_m(A) \qquad (70)$$

Multiply eqn (70) with

$$\left[\frac{p_m(A)}{p_{st.l}(A)}\right] dA$$

and integrate from 0 to ∞

$$\int_0^\infty \frac{p_0(A)p_m(A)}{p_{st.l}(A)}\,dA = \int_0^\infty \frac{p_{st}(A)p_m(A)}{p_{st.l}(A)}\,dA$$
$$+ \int_0^\infty \sum_{m=1}^{n}\sum_{k=1}^{n} D_k K_m^{(k)} \frac{p_m(A)p_m(A)}{p_{st.l}(A)}\,dA \qquad (71)$$

where $p_{st.l}(A)$ is a stationary density distribution function of probability for a linear system obtained from eqn (66) under the assumption that m = 0, $L_0(\ldots)$ = 1.

Taking into consideration the orthogonality of functions $p_{st.l}(A)$ and eqn (66), we obtain:

$$\int_0^\infty \frac{p_m(A)p_m(A)}{p_{st.l}(A)}\,dA = \int_0^\infty \frac{A}{\sigma_1^2}\exp\left(-\frac{A^2}{2\sigma_1^2}\right)$$
$$\times L_m^2\left(\frac{A^2}{2\sigma_1^2}\right) dA = 1 \qquad (72)$$

From eqns (71) and (72), the equations for the coefficients D_k is calculated as:

$$\sum_{k=1}^{n} D_k K_m^{(k)} = \int_0^\infty p_0(A)L_m\left(\frac{A^2}{2\sigma_1^2}\right) dA$$
$$- \int_0^\infty p_{st}(A)L_m\left(\frac{A^2}{2\sigma_1^2}\right) dA \qquad (73)$$

Thus the nonstationary density distribution of the amplitude probability is obtained:

$$p(A,t) = p_{st}(A) + \sum_{m=1}^{n}\sum_{k=1}^{n} D_k K_m^{(k)} e^{\lambda_k t} p_m(A) \qquad (74)$$

The value of $p(A, t)$ can be calculated with any degree of accuracy independent of the number n. The density

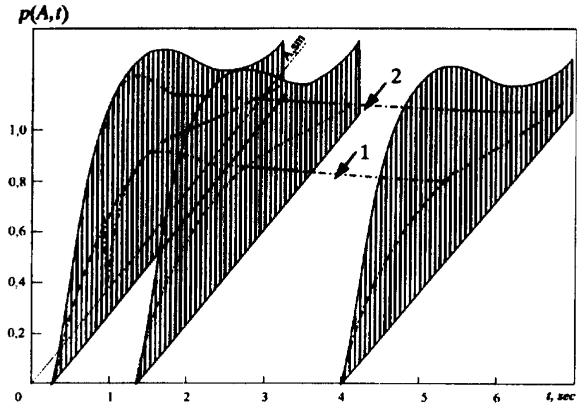

Fig. 2. Probability density function of a nonlinear parametric system in nonstationary regime.

distribution function of probability $p(A, t)$ provides the statistical evolution of vibration amplitude of eqn (56) and takes into account the transient processes. When $t \to \infty$, $p(A, t) \to p_{st}(A)$, which is the solution for the steady-state vibration. The density distribution function of probability $p(A, t)$ gives us exhaustive information about the statistical behavior of the amplitude as a main characteristic of the motion process. On the basic of the function $p(A, t)$, it is possible to evaluate the statistical moments of vibration amplitude using the fundamental formulas of probability theory and evaluate the probability for the amplitude to exceed a concrete level. For example, it is possible to evaluate stresses in machines and structures to assuare safety.[16,17,20,22,23]

4.1 Examples: statistical analysis of nonlinear systems with stochastic parametric excitations as a benchmark application of the FPK equation

Consider a special case of eqn (56) when $\gamma_1 = \gamma_2 = \varepsilon = 0$.

Example 2. This relates to nonlinear system with nonlinear inertia and stochastic linear parametric excitations. In this case the equation of motion is written as following:

$$\ddot{x} + [2\beta_0 + 2\mu\alpha(t)]\dot{x} + \Omega^2[1 + 2\sigma\chi(t)]x + 2k(\ddot{x}x + (\dot{x})^2) = -\nu\eta(t) \qquad (75)$$

The expression by a coefficient k describes the nonlinear inertia of eqn (56). Taking into account eqns (63) and (65) using the second approach, we obtain:

$$p_{st}(A) = \frac{CA}{(a_7 + a_8^*)^b}\exp\left\{\frac{l_1A^2}{a_8^*}\right\} \qquad (76)$$

and

$$p(A,t) = p_{st}(A) + \left\{\frac{1}{2\sigma_1^2}[\langle A^2\rangle - \langle A_0^2\rangle]p_1(A)\exp(\lambda_1, t)\right\} + \left\{\frac{1}{2\sigma_1^2}[\langle A^2\rangle - \langle A_0^2\rangle] + \frac{1}{8\sigma_1^4}[\langle A_0^4\rangle - \langle A^4\rangle]p_2(A)\exp(\lambda_2 t)\right\} \qquad (77)$$

where

$$b = \frac{a_1a_8^* + l_1a_7}{(a_8^*)^2};\ l_1 = -\frac{k\beta_0}{2};\ \lambda_1 = -\frac{\beta_{11}}{\alpha_{11}} = -0.13;$$

$$\lambda_2 = -0.18;\ p_1(A) = \frac{a}{\sigma_1^2}\exp\left\{-\frac{A^2}{2\sigma_1^2}\right\}\left[1 - \frac{A^2}{2\sigma_1^2}\right];$$

$$p_2(A) = \frac{A}{\sigma_1^2}\exp\left\{-\frac{A^2}{2\sigma_1^2}\right\}\left[1 - \frac{A^2}{2\sigma_1^2} + \frac{A^4}{8\sigma_1^2}\right] \qquad (77a)$$

For $k = 0$,

$$p_{st}(A) = \frac{CA}{(a_7 + a_8^*A^2)^{b_0}};\ b_0 = \frac{a_1}{a_8^*} \qquad (78)$$

Following this, the expression for $p(A, t)$ in this case is similar to eqn (77).

For $\mu = 0$, we obtain from eqn (76):

$$p_{st}(A) = \frac{CA}{(a_7 + a_8A^2)^{b_1}}\exp\left\{\frac{l_1A^2}{a_8}\right\};\ b_1 = \frac{a_1}{a_8} \qquad (79)$$

and for $k = 0$ eqn (79) becomes

$$p_{st}(A) = \frac{CA}{(a_7 + a_8A^2)^{b_1}} \qquad (80)$$

Fig. 2 shows the evolution of $p(A, t)$ in the time domain. Curve 1 describes the behavior of the nonlinear system according to eqns (76) and (77) and curve 2 describes the behavior of the linear system according to eqns (77) and (78). This graphical representations of simulation results demonstrated dynamic characteristics of the system in the transient process, the influence of nonlinearity on the dynamic behavior, the sensivity of the dynamic motion to the parametric excitations, and so on. Using eqns (76) and (77), it is possible to perform such tasks as safety valuation, stochastic stability, and optimization of structure.[16–23]

Fig. 3 shows the time histories of the first three statistical moments calculated by formulas eqns (76) and (77) taking into account the transient process. The third statistical moments for the nonlinear system have nonzero values; moreover, for a low-frequency system the third statistical moment can be negative in the transient period of motion. This fact characterizes a deviation from a normal law and shows an essential peculiarity of nonlinear systems. In Part 2 of this paper, we took into consideration this result in the models of statistical linearization of nonlinear stochastic dynamic systems.

There is an essential interest in studying stationary forced vibrations of nonlinear systems. Let us briefly discuss a special case of eqn (63).

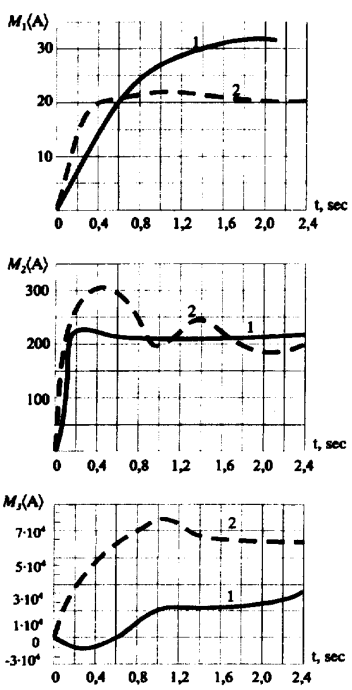

Fig. 3. Statistical moments M_1, M_2 and M_3. 1: $\omega = 20$; 2: $\omega = 60$.

Example 3. In the case of nonlinear inertia when $\alpha(t) = \chi(t) = 0$ in eqn (75), a 'shortened equation' for the amplitude A from eqn (60) is written as:

$$\dot{A} = -a_1 A + I_1 A^3 + m_1 + \xi_1(t);$$

$$m_1 = \frac{\nu^2}{4\Omega^2 A} S\eta(\Omega);\ K_1 = \frac{\nu^2}{2\Omega^2} S\eta(\Omega);\ I_1 = \frac{k\beta_0}{2} \tag{80a}$$

Then:

$$p_{st}(A) = \frac{A}{C}\exp\left[\frac{2}{K_1}\left(-\frac{a_1}{2}A^2 + \frac{I_1}{4}A^4\right)\right] \tag{80b}$$

The constant C is evaluated from eqn (64).

As an example, for a pendulum with a length l, a physically permissible amplitude of vibration does not exceed the value $l/2$. Thus:

$$C = \frac{(l/2)^2}{2}\int_0^l \exp[c_1(-z + c_2 z)]\,dz;$$

$$c_1 = \frac{a_1(l/2)^2}{K_1};\ c_2 = \frac{I_1(l/2)^2}{2a_1} \tag{80c}$$

Changing the upper bound of integration in eqn (64), the integral diverges by $A \to \infty$. Whereas from physical point of view, the value of vibration amplitude $A \ll l/2$. If the nonlinear inertia is absent ($k = 0$), then $I_1 = 0$ and for a linear system we obtain the Rayleigh distribution.

Example 4. Consider a stationary regime of the forced vibration of a nonlinear system [eqn (56)] with the nonlinear inertia ($k > 0$), elasticity ($\gamma_1 < 0$, $\gamma_2 < 0$) and damping

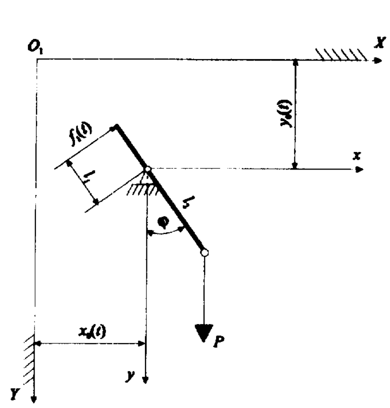

Fig. 4. Physical pendulum.

($\varepsilon > 0$). In this case for $\mu = \sigma = 0$ and eqn (63) rewritten:

$$p_{st}(A) = \frac{A}{C}\exp\left[-\frac{2}{K_1}\left(\frac{a_1}{2}A^2 + \frac{I_1}{4}A^4 + \frac{I_2}{6}A^6 + \frac{I_3}{8}A^8\right)\right] \tag{80d}$$

where

$$a_1 = \beta_0;\ I_1 = \frac{\varepsilon}{8} - \frac{\beta_0 k}{2};\ I_2 = \frac{\varepsilon\gamma_1}{32\Omega^2};\ I_3 = \frac{5\varepsilon\gamma_1}{256\Omega^2};$$

$$C = \frac{K_1}{2a_1}\int_0^\infty \exp[-(z + c_3 z^2 + c_4 z^3 + c_5 z^5)]\,dz;$$

$$c_3 = \frac{I_1 K_1}{2a_1^2};\ c_4 = \frac{I_2 K_1}{3a_1^3};\ c_5 = \frac{I_3 K_1}{4a_1^4} \tag{80c}$$

In this case, the integral for evaluating for a constant C converges. If $k = 0$ then $I_1 = \varepsilon/8$ and the solution corresponds to a nonlinear elastic system.[18,23,28]

Example 5. This pertains to a physical pendulum with stochastic nonlinear parametric excitations. Figure 4 shows the model of a physical pendulum. The coordinate system (X, O_1, Y) is an inertial coordinate system, while the coordinate system (x, o, y) is connected with the pendulum fulcrum, translating with a law $x_0(t)$ and $y_0(t)$. A force $f_1(t)$ acts on the pendulum, as shown in Figure 4.

The model of a physical pendulum provides an important description for: (1) the vibration of a rigid structure subjected to seismic ground motion in vertical and horizontal directions; and (2) vibrations of a parachute with a load on a trajectory relative to the center of mass. In the first case, components of the ground motion are parametric excitations, while in second case, horizontal and vertical air flows are parametric excitations. Because this model has a broad engineering application, the solution of a corresponding FPK equation is studied as a benchmark.

Assuming a viscous damping force, the differential equation of motion for the pendulum can be written as:[16]

$$I_z\ddot{\varphi} = -b\dot{\varphi} - Pl_2\sin\varphi - M\ddot{x}_0(t)l_2$$

$$\times\cos\varphi - M\ddot{y}_0(t)l_2\sin\varphi - l_1 f_1(t)$$

where φ is the angle of the pendulum, I_z is moment of inertia of the pendulum with respect to axis z passed through point O; M is a pendulum mass.

The functions $\ddot{x}_0(t)$, $\ddot{y}_0(t)$ and $f_1(t)$ are correlated stationary random functions of time with fixed probability characteristics (with known density probability functions, autocorrelation functions and power spectral density functions with zero means).

Rewrite eqn (81) as:

$$\ddot{\varphi}+2\beta_0\dot{\varphi}+\Omega^2\sin\varphi+\mu_2 f_2(t)\cos\varphi+\mu_3 f_3(t)\sin\varphi = -\mu_1 f_2(t) \qquad (82)$$

where

$$2\beta_0=\frac{b}{I_z};\ \Omega^2=\frac{Pl_2}{I_z};\ \mu_1=\frac{Ml_1}{I_z};\ \mu_2=\mu_3=\frac{Ml_2}{I_z};$$

$$f_2(t)=\ddot{x}_0(t)\ f_3(t)=\ddot{y}_0(t) \qquad (82a)$$

This class of mathematical model is a generalization for a stochastic variant of a dynamic system described by such a differential equation as the Mathieu–Hill type equations.[18–20]

Specific qualities of stochastic nonlinearity were discussed in Refs[16–20,24,25]. The above-mentioned peculiarity is essential for a choice of a method for statistical analysis and a stability criterion for the dynamic systems.[16,18–20]

Expanding the trigonometric functions into power series and taking into account first two terms under the assumption that φ is small, eqn (82) can be rewritten as:

$$\ddot{\varphi}+2\beta_0\dot{\varphi}+\Omega^2[1+\mu f(t)]\varphi-\frac{1}{2}\mu_2 f_2(t)\varphi^2-\frac{1}{6}\mu_3 f_3(t)\varphi^3 - \gamma_0\varphi^3=\nu\eta(t) \qquad (83)$$

where

$$\gamma_0=\frac{\Omega^2}{6};\ \mu=\frac{\mu_3}{\Omega^2};\ \mu_1 f_1(t)+\mu_2 f_2(t)=\nu\eta(t);\ f_3(t)=f(t) \qquad (83a)$$

The equation for amplitudes of vibration with extracted vibrational functions from the regular and fluctuation terms in the second approach is:

$$\dot{A}=-\beta_0 A+m_1+\xi_1(t) \qquad (84)$$

where

$$m_1=\frac{1}{9216}\frac{\mu_3^2}{\Omega^2}A^5[4S_{f_3}(2\Omega)+S_{f_3}(4\Omega)]+\frac{1}{32}\frac{\mu_2^2}{\Omega^2}A^3 \times[S_{f_2}(\Omega)+S_{f_2}(3\Omega)]-\frac{1}{12}\mu\mu_3A^3S_{ff_3}(2\Omega) +\frac{5}{16}\frac{\mu_2\nu}{\Omega^2}AS_{f_2\eta}(\Omega)-\frac{5}{8}\Omega^2\mu^2AS_f(2\Omega)+\frac{\nu^2}{2\Omega^2A^2}S_\eta(\Omega) \qquad (85)$$

The coefficient of intensity for a process $\xi_1(t)$ as:

$$K_1=\int_{-\infty}^{\infty}\langle\xi_1(t)\xi_1(t+\tau)\rangle\,d\tau=\frac{1}{2}\frac{1}{4608}\frac{\mu_3^2}{\Omega^2}A^6 \times[S_{f_3}(2\Omega)+S_{f_3}(4\Omega)]+\frac{1}{128}\frac{\mu_2^2}{\Omega^2}A^4 \times[S_{f_2}(\Omega)+S_{f_2}(3\Omega)]-\frac{1}{24}\mu\mu_3A^4S_{ff_3}(2\Omega) +\frac{1}{8}\frac{\mu_2\nu}{\Omega^2}A^2S_{f_2\eta}(\Omega)-\frac{1}{2}\Omega^2\mu^2A^2S_f(2\Omega)+\frac{\nu^2}{2\Omega^2}S_\eta(\Omega) \qquad (86)$$

The functions m_1, $\xi_1(t)$ are calculated by applying harmonic linearization.

In eqns (85) and (86), the functions $S_{f_3}(2\Omega)$ and $S_{f_3}(4\Omega)$ are power spectral densities of a process $f_3(t)$ on the second and fourth frequencies; $S_{f_2}(\Omega)$ and $S_{f_2}(3\Omega)$ are power spectral densities of the process $f_2(t)$ on the first and third frequencies; $S_{ff_3}(2\Omega)$ is a cross-spectral density function between of processes $f(t)$ and $f_3(t)$ on the second frequency; $S_{f_2\eta}(\Omega)$ is cross-spectral density (a cross-correlation function defined) of processes $f_2(t)$ and $\eta(t)$ on the first frequency; $S_f(2\Omega)$ is power spectral density function of a process $f(t)$ on the second frequency; $S_\eta(\Omega)$ is power spectral density function of a process $\eta(t)$ on the first frequency.

The FPK equation for the probability density function of eqn (84) is:

$$\frac{\partial p(A,t)}{\partial t}=-\frac{\partial}{\partial A}\{[a_7A^{-1}+5(a_5-a_6-a_1)A +2(a_3-a_4)A^3+a_2A^5]p(A,t)\} +\frac{1}{2}\frac{\partial^2}{\partial A^2}\{[(a_7+2(a_5-2a_6)A^2+(a_3+a_4)A^4 +2a_2A^6]p(A,t)\} \qquad (87)$$

where

$$a_1=\frac{1}{5}\beta_0;\ a_2=\frac{\mu_3}{9216}[S_{f_3}(2\Omega)+S_{f_3}(4\Omega)];$$

$$a_3=\frac{1}{128}\frac{\mu_2^2}{\Omega^2}[S_{f_2}(\Omega)+S_{f_2}(3\Omega)];\ a_4=\frac{1}{24}\mu\mu_3S_{ff_3}(2\Omega);$$

$$a_5=\frac{1}{16}\frac{\mu_2\nu}{\Omega^2}S_{f_2\eta}(\Omega);\ a_6=\frac{1}{8}\Omega^2\mu^2S_f(2\Omega);\ a_7=\frac{\nu^2}{2\Omega^2}S_\eta(\Omega) \qquad (88)$$

The stationary distribution of probability for eqn (87) is:

$$p_{st}(A)=\frac{C}{[(a_7+2(a_5-2a_6)A^2+(a_3+a_4)A^4+2a_2A^6]} \times\exp\left\{2\int\frac{a_7A^{-1}+5(a_5-a_6-a_1)A+2(a_3-a_4)A^3+a_2A^5}{a_7+2(a_5-2a_6)A^2+(a_3+a_4)A^4+2a_2A^6}\,dA\right\} \qquad (89)$$

The constant C is evaluated from:

$$\int_0^{2\pi}p_{st}(A)\,dA=1 \qquad (90)$$

The upper limit of integral eqn (90) depends on the physical condition.

In this case:

$$C=\frac{1}{\frac{1}{2a_2}\left[\frac{(2\pi)^2}{2}D+\frac{(2\pi)^4}{4}D_1+\frac{(2\pi)^6}{6}D_2+\frac{(2\pi)^8}{8}D_3\right]} \quad (91)$$

where

$$D=\alpha^{(B_1-1)}\beta^{-(B_2+1)}\gamma^{(B_3-1)};$$

$$\begin{aligned} D_1 &= \alpha^{(B_1-1)}\beta^{-(B_2+1)}\gamma^{(B_3-1)}(B_2+1) \\ &\quad -\alpha^{(B_1-2)}\beta^{-(B_2+1)}\gamma^{(B_3-1)}(B_1-1) \\ &\quad -\alpha^{(B_1-1)}\beta^{-(B_2+1)}\gamma^{(B_3-2)}(B_3-1); \\ D_2 &= \alpha^{(B_1-1)}\beta^{-(B_2+1)}\gamma^{(B_3-2)}(B_1-1)(B_3-1) \\ &\quad -\alpha^{(B_1-1)}\beta^{-(B_2+2)}\gamma^{(B_3-2)}(B_2+1)(B_3-1) \\ &\quad -\alpha^{(B_1-2)}\beta^{-(B_2+2)}\gamma^{(B_3-1)}(B_1-1)(B_2+1) \\ D_3 &= \alpha^{(B_1-2)}\beta^{-(B_2+2)}\gamma^{(B_3-2)}(B_1-1)(B_2+1)(B_3-1) \end{aligned} \quad (91a)$$

and

$$\begin{aligned} B_1 &= \frac{a_5[\alpha^2-\alpha^2\beta+\beta^2+\gamma]}{2a_2\alpha(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)} \\ &\quad -\frac{(a_1-2a_4)(\gamma-\beta)}{2a_2(\alpha-\gamma)[\beta(\beta-\alpha)-(\gamma-\alpha)]} \\ &\quad -\frac{5a_3\alpha}{2a_2(\alpha-\beta)(\alpha-\gamma)}+\frac{7\alpha^2}{2(\alpha^2-\alpha-\beta\gamma)}; \\ B_2 &= \frac{5a_3\beta}{2a_2(\alpha-\beta)(\gamma-\beta)}-\frac{7}{2}\frac{\beta[\alpha-\gamma(\alpha+\beta)]}{(\beta-\gamma)[\alpha^2-\alpha-\beta\gamma]} \\ &\quad -\frac{a_5}{2a_2\beta(\alpha-\beta)(\beta-\gamma)}-\frac{(a_1-2a_4)}{2a_2[\beta(\beta-\alpha)-(\gamma-\alpha)]}; \\ B_3 &= \frac{5a_3\gamma}{2a_2(\gamma-\alpha)(\gamma-\beta)}+\frac{7}{2}\frac{\gamma[\alpha-\beta(\alpha+\beta)]}{(\beta-\gamma)[\alpha^2-\alpha-\beta\gamma]} \\ &\quad +\frac{a_5}{2a_2\gamma(\alpha-\beta)(\beta-\gamma)} \\ &\quad -\frac{(a_1-2a_4)(\beta-\alpha)}{2a_2(\alpha-\gamma)[\beta(\beta-\alpha)-(\gamma-\alpha)]}; \end{aligned} \quad (91b)$$

$$\alpha=-\frac{2}{a_1}k_1\cos\left(\frac{1}{3}\arccos k_2\right)-k_3;$$

$$\beta=\frac{1}{a_7}k_1\left\{\cos\left(\frac{1}{3}\arccos k_2\right)\right.$$

$$\left.+\sqrt{3}\left[1-\cos^2\left(\frac{1}{3}\arccos k_2\right)\right]^{1/2}\right\}-k_3$$

The coefficients γ, k_1, k_2 and k_3 define as:

$$\gamma=\frac{1}{a_7}k_1\left\{\cos\left(\frac{1}{3}\arccos k_2\right)\right.$$

$$\left.-\sqrt{3}\left[1-\cos^2\left(\frac{1}{3}\arccos k_2\right)\right]^{1/2}\right\}$$

$$-k_3;\ k_3=\frac{2(a_6-a_5)}{3a_7};$$

$$k_1=\sqrt{\frac{3a_7(a_3-a_4)-4(a_6-a_5)^2}{6}};$$

$$k_2=\frac{4(a_6-a_5)^3-9(a_6-a_5)(a_3-a_4)a_7-27a_2a_7^2}{27\left(\sqrt{\frac{3a_7(a_3-a_4)-4(a_6-a_5)^2}{6}}\right)^3}$$

The nonstationary solution of eqn (87) is obtained from the Bubnov–Galerkin variational method as:

$$p(A,t)=p_{st}(A)+\left\{\frac{1}{2\sigma_1^2}[\langle A^2\rangle-\langle A_0^2\rangle]p_1(A)\exp(\lambda_1 t)\right\}$$

$$+\left\{\frac{1}{2\sigma_1^2}[\langle A^2\rangle-\langle A_0^2\rangle]+\frac{1}{8\sigma_1^4}[\langle A_0^4\rangle-\langle A^4\rangle]p_2(A)\exp(\lambda_2 t)\right\},$$

$$\sigma_1^2=\frac{a_7}{5(a_5-a_6)} \quad (92)$$

where

$$p_{st}(A)=\frac{CA(A^2-\alpha)^{B_1-1}(A^2-\beta)^{B_2+1}(A^2-\gamma)^{B_3-1}}{2a_2} \quad (93)$$

and

$$\langle A^2\rangle=C\left\{\frac{(2\pi)^4}{4}D+\frac{(2\pi)^6}{6}D_1+\frac{(2\pi)^8}{8}D_2+\frac{(2\pi)^{10}}{8}D_3\right\};$$

$$\langle A^2\rangle=C\left\{\frac{(2\pi)^6}{6}D+\frac{(2\pi)^8}{8}D_1+\frac{(2\pi)^{10}}{10}D_2+\frac{(2\pi)^{12}}{12}D_3\right\};$$

$$p_0(A)=A\sigma_1^{-2}\exp\left(-\frac{A^2}{2\sigma_1^2}\right) \quad (94)$$

Here:

$$p_1(A)=\frac{A}{\sigma_1^2}\exp\left(-\frac{A^2}{2\sigma_1^2}\right)\left(1-\frac{A^2}{2\sigma_1^2}\right);$$

$$p_2(A)=\frac{A}{\sigma_1^2}\exp\left(-\frac{A^2}{2\sigma_1^2}\right)\left(1-\frac{A^2}{2\sigma_1^2}+\frac{A^4}{8\sigma_1^4}\right) \quad (94a)$$

Therefore, a complete statistical description is given for the nonlinear motion of the physical pendulum.

5 CONCLUSIONS

In this paper the probabilistic description of the response of a nonlinear parametric system driven by external stochastic processes is discussed. On the basis of the asymptotic method of nonlinear mechanics, an exact solution of the FPK equation for a probability density function is introduced for a class of nonlinear parametric systems. Using the solution of the FPK equation the transient and stationary processes of this class of dynamic systems are studied as is the sensitivity of the nonlinear system to different stochastic color correlated parametric excitations. It is shown that the dynamic effect of the nonstationarity on the response may be significant and the peak amplitude of the nonstationary moment response may be higher than that in the stationary case. Moreover, the third statistical moment has nonzero values and different behaviors can be positive or negative for essential nonlinear dynamic systems. This result characterizes a deviation of response output from a normal law and shows an essential peculiarity of nonlinear systems. The FPK equation approach provides background for studying stochastic stability of nonlinear stochastic parametric systems with stochastic nonlinearity in Parts 2 and 3 of this paper.

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APPENDIX A GENERALIZED SOLUTIONS OF FPK EQUATIONS

In this appendix, we describe a different methods of a study of the solutions of the FPK equations. Considerable interest has recently been devoted to the determination of the 'generalized symmetries' of a differential equation. We study nonlinear forms of a linear partial differential equations and extended symmetry properties of time-evolution partial differential equations, including the FPK-type equations. Even if (in general) no nontrivial symmetry present, in some a particular interesting cases, some special symmetry is allowed. Using this approach, we can define a new solutions of the FPK equation (as a nonlinear form of any linear equation) on basis of a particular solution of a linear equations.

Appendix A.1 Case A1: Lie group analysis of generalized symmetries and solutions of FPK equations

If a time-evolution of a partial differential equation has a solution $f = f(x, t)$, $x \in R$, $t \in R$, a generalized symmetry is any continuous transformation (possible nonlinear or only local) $x \to x'$, $t \to t'$, $f \to f'(x, t)$ is also a solution of the given equation[61]. These transformations are assumed to depend analytically on a real parameter ε, so that attention is mainly centered on their Lie generators, which, in this case, can be written in the general form[61,62] as:

$$v = \xi(x, t, w)\frac{\partial}{\partial x} + \tau(x, t, w)\frac{\partial}{\partial x} + \phi(x, t, w)\frac{\partial}{\partial w} \qquad \text{(A1)}$$

where ξ, τ, ϕ are the functions to be determined, and

$$V = g(x, t, w, w_1, \dots, w_n)\frac{\partial}{\partial w},$$

$$w = \frac{\partial^i}{\partial w_i},\ w_i = \frac{\partial^i w}{\partial x^i}\,(i = 1, \dots, n) \qquad \text{A2)}$$

is a hierarchy of a time-dependent Lie–Bäcklund vector field.[61] The assumption that the analytic function g depends also on x and t does not affect the results. This means the existence of a Lie– Bäcklund vector field and therefore, for the sake of simplicity, they will be omitted.

We investigate the symmetry properties of one-variable FPK-type equation, namely the equations of the form:

$$\frac{\partial w}{\partial t} = -\frac{\partial}{\partial x}(a(x)w) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(g^2(x)w) \qquad \text{(A3)}$$

or also in the more general form

$$\frac{\partial w}{\partial t} = A(x)w + B(x)\frac{\partial w}{\partial x} + C(x)\frac{\partial^2 w}{\partial x^2} \qquad \text{(A4)}$$

where a, g or, respectively, A and B, are given the regular (analytical) functions of $x \in R$, with $C(x) = 1/2g(x) \neq 0$.

For describing the Lie–Bäcklund vector fields evolution equation, the jet bundle technique is a suitable approach.[61] We consider the submanifold:

$$F \equiv \frac{\partial w}{\partial t} - A(x)w - B(x)\frac{\partial w}{\partial x} - C(x)\frac{\partial^2 w}{\partial x^2} = 0 \qquad \text{(A5)}$$

and all its differential consequences with respect to the space coordinate.

The invariance requirement is expressed as:

$$L_V F \overset{d}{=} 0 \tag{A6}$$

where $L_V(.)$ denotes the Lie derivative and $\overset{d}{=}$ stands for the restriction to the solutions of eqn (A5).

We construct the second prolongation of the vector field eqn (A1) and apply it to eqn (A3) or eqn (A4) in order to find the conditions on the functions ξ, τ and ϕ, in such a way that eqn (A1) generates a symmetry of eqns (A3) and (A4). For eqn (A4), the conditions are following:

$$\tau_x = \tau_w = 0,\ \xi_w = 0,\ \phi_{ww} = 0, \tag{A7}$$

implying $\tau = \tau(t)$, $\xi = \xi(x,t)$ only, and

$$\phi = \alpha(x,t) + \beta(x,t)w,\ \xi_x = \xi\frac{C_x}{2C} + \frac{\tau_t}{2} \tag{A8}$$

This equation can be solved with respect to x to get

$$\xi = c(t)g + \frac{1}{2}\tau_t g G \tag{A9}$$

where $c = c(t)$ is a function of t and $G = G(x)$ is an integral function of $g^{-1}(x)$. Accordingly[62] we have:

$$\beta_x = p_1 c + p_2 c_t + p_3 \tau_t + p_4 \tau_{tt},\ p_i = p_i(x) \tag{A10}$$

and

$$\beta_t = q_1 c + q_2 c_t + q_3 \tau_t + q_4 \tau_{tt},\ q_i = q_i(x) \tag{A11}$$

where

$$p_1 = \frac{Bg_x}{g^2} - \frac{B_x}{g} + \frac{g_{xx}}{2};\ p_2 = -\frac{1}{g};\ p_3 = -\frac{Bg_xG}{2g^2} - \frac{B}{2g^2} - \frac{B_xG}{2g} + \frac{g_{xx}G}{4} + \frac{g_x}{4g};\ p_4 = -\frac{G}{2g} \tag{A12}$$

and

$$q_1 = Bp_1 + \frac{g^2 p_{1x}}{2} + A_x g;\ q_2 = Bp_2 + \frac{g^2 p_{2x}}{2};$$
$$q_3 = Bp_3 + \frac{g^2 p_{3x}}{2} + \frac{A_x g G}{2} + A;\ q_4 = Bp_4 + \frac{g^2 p_{4x}}{2} \tag{A13}$$

In order to find β from eqns (A10) and (A11) we have now to impose the condition $\beta_{tx} = \beta_{xt}$, which becomes:

$$c_{tt} + g q_{1x} c = -\left(\frac{1}{2}G\tau_{ttt} + g q_{3x}\tau_t\right) \tag{A14}$$

If there is no a special relationship between the functions G, gq_{1x}, gq_{3x} appearing in eqn (A14), the only solution allowed by eqn (A14) is just $c = 0$, $\tau_t = 0$, which leads precisely to the 'trivial' symmetries generated by $v_1 = \partial/(\partial t)$, $v_2 = w(\partial)/(\partial w)$, $v_3 = \alpha(\partial)/(\partial w)$ [if w and α are solutions of eqns (A3) and (A4), the same is true for $kw + \alpha$] shared by all autonomous linear equations.

Some nontrivial symmetries can arise if some relationship occurs between the functions of x in eqn (A14). If in eqn (A14), assume $a = x$, $g = 1$ and then from eqn (A14) becomes:

$$c_{tt} - c = -\frac{1}{2}x(\tau_{ttt} - 4\tau_t),\ c_{tt} = c,\ \tau_{ttt} = 4\tau_t \tag{A15}$$

and the new symmetries generated[62,63] by

$$v_4 = e^t\frac{\partial}{\partial x};\ v_5 = e^{-t}\frac{\partial}{\partial x} + 2xe^{-t}w\frac{\partial}{\partial w};$$
$$v_6 = xe^{2t}\frac{\partial}{\partial x} + e^{2t}\frac{\partial}{\partial t} - e^{2t}w\frac{\partial}{\partial w};$$
$$v_7 = xe^{-2t}\frac{\partial}{\partial x} - e^{-2t}\frac{\partial}{\partial t} + 2x^2e^{-2t}w\frac{\partial}{\partial w} \tag{A15a}$$

are present. In this case, v_4 expresses the property that if $w(x,t)$ solves the equation, then, in addition, $w(x + \varepsilon e^t, t)$, $\varepsilon \in R$, does; v_5 states also that $\exp(2\varepsilon x e^{-t} + \varepsilon^2 e^{-2t})\, w(x + \varepsilon e^{-t}, t)$ is a solution, whereas the remaining two operators generate more complicated symmetries involving simultaneous x, t, w. For a case $a = x$, $g = x$ from eqn (A14), it follows $c_{tt} = \tau_{ttt} = 0$ and

$$v_4 = x\frac{\partial}{\partial x};\ v_5 = xt\frac{\partial}{\partial x} - \left(\ln x + \frac{t}{2}\right)w\frac{\partial}{\partial w};$$
$$v_6 = \frac{x}{2}\ln x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{1}{4}\left(\ln x + \frac{t}{2}\right)w\frac{\partial}{\partial w};$$
$$v_7 = xt\ln x\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} - \frac{1}{2}\left(t\ln x + \ln^2 x + t + \frac{t^2}{4}\right)w\frac{\partial}{\partial w} \tag{A15b}$$

In this case v_4 expresses the rather elementary property of a scale invariance of the equation.

Example A1. The equation

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + f(w) \tag{A16}$$

belongs to the the following class of the partial differential equations which admit a hierarchy of Lie–Bäcklund vector fields[64] as eqn (A2):

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + f_1(w)\left(\frac{\partial w}{\partial x}\right)^2 + f_2(w)\frac{\partial w}{\partial x} + f_3(w) \tag{A17}$$

and the functions f_1, f_2 and f_3 satisfy the system of the differential equations

$$f'_2 f_3 = 0,\ f'_2 f_1 = f''_2,\ f'''_3 + (f_1 f_3)' = 0 \tag{A17a}$$

Thus, if $f_3 = f_1 = 0$ and $f_2 = u$, then eqn (A17a) is satisfied and we obtain the well-known Burgers equation. Like the Lie point vector fields, the Lie–Bäcklund vector fields as eqn (A2) can also be used for finding solutions to the underlying partial differential equation.[61]

Appendix A.2 Case A2: Nonlinear forms and transformations of linear equations as solutions of FPK equations

For a study a solutions of nonlinear equations as:

$$\frac{\partial w}{\partial t} = \frac{\partial^l w}{\partial x^l} + \varepsilon F_2\left(w, \frac{\partial w}{\partial x}, \ldots\right) + \ldots + \varepsilon^{r-1} F_r\left(w, \frac{\partial w}{\partial x}, \ldots\right) \tag{A18}$$

where F_s is a polinom of sth degree. Discuss solutions of a linear equation:

$$\frac{\partial u}{\partial t}=\frac{\partial^l u}{\partial x^l} \tag{A19}$$

and define *a transformation operator* $w = Ku$ that transform a solutions of eqn (A19) in a solutions of eqn (A18). The solutions of eqn (A18) define[65] as:

$$w(x)=u(x)+\sum_{n=2}^{\infty}\varepsilon^{n-1}\int K_n(p_1,\ldots,p_n)u(x+p_1)\ldots u(x+p_n)\,\mathrm{d}p_1\ldots\,\mathrm{d}p_n \tag{A20}$$

where

$$K_n(p_1,\ldots,p_n)=k_n(\partial_1,\ldots,\partial_n)\delta(p_1)\ldots\delta(p_n),\ \partial_i=\frac{\partial}{\partial p_i} \tag{A21}$$

and for definition of coefficients k_n exist a system of the algebraic recurrence formulas[65]

$$[(\partial_1+\ldots+\partial_n)^l-\partial_1^l-\ldots-\partial_n^l]k_n=\Phi_n(\partial_1,\ldots,\partial_n),\ k_1=1 \tag{A22}$$

In this case, eqn (A18) be called the *local integrable equation* if $k_n=P_n(\partial_1,\ldots,\partial_n)/Q(\partial_1,\ldots,\partial_n)$, where P_n, Q_n polinoms and Q_n decompose on a linear factors.

Example A2. For an equation $w_t = (1+\varepsilon w)^2 w_{xx}$, we have any generalization as:

$$w_t=\left(\frac{1+ap(w)}{1+w_xq'(w)}\right)^2(w_{xx}-q''(w)w_x^3)+\frac{p''(w)(1+ap(w))^2}{p'(w)(1+w_xq'(w))}w_x^2+bp(w)(2+ap(w))w_x \tag{A22a}$$

where p, q are any functions ($p'\neq 0$) and a, b are any constants. Transformation operators as:

$$p(w)=\frac{u_y}{1+bu},\ y-q(w)+\frac{a}{b}\ln(1+bu)=x \tag{A22b}$$

define[65] a solutions of this equations from a solutions of the heat equation $u_t = u_{yy}$.

The method of a study any nonlinear forms of a linear equations for the equations as $a\psi_t+b\psi_{xt}+c\psi_{tt}+d\psi_{xx}=0$ (where a, b, c and d are any functions of a variables x, t) after a substitute $\psi=\exp(\gamma w+\delta w_x+\beta_t)$ give three types of a nonlinear partial differential equations[66] relative to first and second derivatives in dependence from a relationship between a coefficients a, b, c, d, γ, δ and β. One of these equations is:

$$a(\gamma w_t+\beta w_{tt})+b(\gamma^2w_x^2+\beta^2w_{tx}^2+2\gamma\beta w_{tx}w_x+\gamma w_{xx}+\beta_{txx})=0 \tag{A23}$$

After transformation:

$$u=\exp(\gamma w+\beta w_t) \tag{A24}$$

we can write a linear equation as

$$au_t+bu_{xx}=0 \tag{A25}$$

Motion integrals[66] for eqn (A25) are:

$$\tilde I_1=\frac{\partial}{\partial t},\ \tilde I_2=\frac{\partial}{\partial x},\ \tilde I_3=x-2tba^{-1}\frac{\partial}{\partial x} \tag{A26}$$

New solutions $\tilde w$ of eqn (A23) can be define from any certain solution w using a transformation:

$$\tilde w=\left(\gamma+\beta\frac{\partial}{\partial t}\right)^{-1}\ln f(\tilde I_1,\tilde I_2,\tilde I_3)\exp\left(\gamma w+\beta\frac{\partial w}{\partial t}\right) \tag{A27}$$

Example A3. For a certain function f in eqn (A27) as:

$$f(\tilde I_1,\tilde I_2,\tilde I_3)=\tilde I_2-\tilde I_1\tilde I_3 \tag{A28}$$

we can define[66] a new solution as

$$\tilde w=\left(\gamma+\beta\frac{\partial}{\partial t}\right)^{-1}[\gamma w+\beta w_t+\ln\{(\gamma w_t+\beta w_{tt})-2ba^{-1}(\gamma w_x+\beta w_{tx}+t[(\gamma w_t+\beta w_{tt})(\gamma w_x+\beta w_{xx})+\gamma w_{xt}+\beta w_{txt}])\}] \tag{A29}$$

Example A4. The general solution of the FPK equation for dissipation in phase-sensitive reservoirs:

$$\frac{\partial w}{\partial t}=\frac{\gamma}{2}\left[\frac{\partial(xw)}{\partial x}+\frac{\partial(x^*w)}{\partial x^*}+M\frac{\partial^2w}{\partial x^2}+2D\frac{\partial^2 w}{\partial x\partial x^*}+M^*\frac{\partial^2w}{\partial x^{*2}}\right] \tag{A30}$$

is given in Ref.[67]. The stationary solution reads:

$$w_{st}=\frac{1}{\pi\sqrt{D^2-|M|^2}}\exp[\frac{M^*x^2-2Dxx^*+Mx^{*2}}{2(D^2-|M|^2)}] \tag{A31}$$

The general solution of eqn (A30) is given by:

$$w=\int w_a(\mathrm{e}^{-(\gamma/2)t}x-\sqrt{1-\mathrm{e}^{-\gamma t}}\beta)w_{st}\times(\sqrt{1-\mathrm{e}^{-\gamma t}}x+\mathrm{e}^{-(\gamma/2)t}\beta)\,\mathrm{d}\beta \tag{A32}$$

By means of the transformation $x_0\equiv\mathrm{e}^{-(\gamma/2)t}x-\sqrt{1-\mathrm{e}^{-\gamma t}}\beta$ we obtain:

$$w=\int G(x,x_0,t)w_a(x_0)\,\mathrm{d}x_0 \tag{A32a}$$

where

$$G(x,x_0,t)=\frac{1}{1-\mathrm{e}^{-\gamma t}}w_{st}(\tilde x),\ \tilde x=\frac{x-x_0\mathrm{e}^{-(\gamma/2)t}}{\sqrt{1-\mathrm{e}^{-\gamma t}}} \tag{A32b}$$

gives the Green's function. For $t=0$ we get:

$$w(x,t=0)=\int w_a(x)w_{st}(\beta)\,\mathrm{d}\beta=w_a(x) \tag{A32c}$$

since w_0 is normalized.

Appendix A.3 Case A3: Principle of superposition for solution of nonlinear partial differential equations

The theory of a linear equations on a semimodular lattice is described in Ref. [68]. On basis of this theory, it is possible to introduce a new principle of superposition of nonlinear equations solution. For example, consider the heat equation:

$$\frac{\partial u}{\partial t}=\frac{h}{2}\frac{\partial^2 u}{\partial x^2},\ x\in R,\ t>0 \tag{A33}$$

where $h > 0$ is any parameter. Linear combination

$$u=\lambda_1 u_1+\lambda_2 u_2 \tag{A33a}$$

of the solutions u_1, u_2 for eqn (A33) is also a solution of eqn (A33). If introduced for eqn (A33), the nonlinear transformation:

$$u=\exp\left\{-\frac{w(x,t)}{h}\right\} \tag{A33b}$$

then define the nonlinear equation

$$\frac{\partial w}{\partial t}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2-\frac{h}{2}\frac{\partial^2 w}{\partial x^2}=0 \tag{A34}$$

as the so-called *Burgers* equation [see also a particular case from eqns (A17) and (A17a)]. In this case, a solution $u_i (i = 1, 2)$ of eqn (A33) equivalent to a solution $w_i = -h\ln u_i$ $(i = 1, 2)$ of eqn (A34). Then for a linear combination $u = \lambda_1 u_1 + \lambda_2 u_2$ of a solution of eqn (A33), define a corresponding solution of eqn (A34) as:

$$w=-h\ln\left(\exp\left\{\frac{(-w_1+\mu_1)}{h}+\frac{(-w_2+\mu_2)}{h}\right\}\right),$$

$$\mu_i=-h\ln\lambda_i\ (i=1,\ 2)$$

where $\mu_i = -h \ln \lambda_i$ $(i = 1, 2)$. Thus, eqn (A34) is also a linear equation, but in the function space where a semigroup operation introduced as: (1) operation of the 'sum' $a\oplus b=-h\ln(\exp\{-a/h\}+\exp\{-b/h\})$; and (2) the operation of 'multiplying' $a\odot\lambda = a + \lambda$. Then the substitute $w = -h\ln u$ transfers $0 \to \infty$ and $1 \to 0$. Thus, a semigroup '0' in this new space be ∞:**0** $= \infty$ and a semigroup '1' is a usual 0:**1** $= 0$. The function space with the introducing operations $\oplus$ and $\odot$ and adjoint to this space a zero '0' and a unit '1' is isomorphic to a usual function space with a usual operations of multiplying and summation.[68]

In function space with values in a ring with the operations:

$$a\oplus b=-h\ln\left(\exp\left\{\frac{-a}{h}\right\}+\exp\left\{\frac{-b}{h}\right\}\right),$$

$$\lambda\odot b=\lambda+b$$

introduce a scalar multiplying

$$(w_1,w_2)=-h\ln\int\exp\left\{-\frac{w_1+w_2}{h}\right\}\mathrm{d}x \tag{A36}$$

This scalar multiplying in this space possesses[68] bilinear properties:

$$(a\oplus b,c)=(a,c)\oplus(b,c),\ (\lambda\odot a,c)=\lambda\odot(a,c) \tag{A37}$$

A self-adjoint operator in this space

$$L:w\to w\odot\left(-h\ln\left(\frac{(w')^2}{h^2}-\frac{w''}{h}\right)\right) \tag{A38}$$

In this case:

$$(w_1,Lw_2)=(Lw_1,w_2) \tag{A39}$$

As a resolving operator for the Burgers equation, define as L_t: $w_0\to w$, where w is a solution of eqn (A34) with initial condition $w|_{t=0}=w_0$. The solution of eqn (A33) with initial condition $u|_{t=0}=u_0$ defines as:

$$u=\frac{1}{\sqrt{2\pi h}}\int\exp\left\{-\frac{(x-\xi)^2}{2+h}\right\}u_0(\xi)\,\mathrm{d}\xi \tag{A40}$$

Then $(w = -h\ln u)$ the resolving operator L_t for the Burgers equation defines as:

$$L_t w_0=-h\ln\left(\frac{1}{\sqrt{2\pi h}}\int\exp\left\{-\frac{(x-\xi)^2}{2+h}+\frac{w_0(\xi)}{h}\right\}\mathrm{d}\xi\right) \tag{A41}$$

The resolving operator L_t is a self-adjoint operator in this space relative to a new scalar multiplying:

$$(w_1,L_t w_2)=(L_t w_1,w_2) \tag{A41a}$$

For $h\to 0$ the Burgers equation, $2w_t+(w_x)^2-hw_{xx}=0$ transfers in the Hamilton–Jacobi equation:

$$\frac{\partial S}{\partial t}+\frac{1}{2}\left(\frac{\partial S}{\partial x}\right)^2=0 \tag{A42}$$

The operation 'summation' $a\oplus b=-h\ln(\exp\{-a/h\}+\exp\{-b/h\})$ for $h\to 0$ transfers in operation $a\oplus b=\min(a, b)$. The operation 'multiplying' independent from h is, therefore, as before: $a\odot\lambda = a + \lambda$.

Appendix A.4 Case A4: The FPK equation and the Schrödinger equations in Nelson's stochastic mechanics

In Nelson's stochastic mechanics the Schrödinger equation is derived from the classical Newtonian mechanics by postulating a particle moving according to a some diffusion processes. The FPK equation for the stochastic process can be interpreted as an equation describing the amplitude of the Schrödinger equation for the wave function. A quantum-mechanical description of a class of stochastic optimal control problems is investigate in Ref.[57]. Two versions of a nonlinear Schrödinger equation are derived. The FPK equation is presented for a diffusion process described by the stochastic differential equation. A stochastic optimal control problem associated with this Markov process, and its Bellman–Hamilton–Jacobi equation for dynamic programming, is studied. A function called the wave function is introduced which combines solutions of the FPK equation and the time-reversal Bellman–Hamilton–Jacobi equation.

This wave function satisfies a nonlinear partial differential equation similar to the Schrödinger-Langevin equation in quantum physics. The key idea stands on the wave-function, defined by Nelson as:

$$\psi(x,t)=\sqrt{p(x,t)}\exp\{iS(x,t)\} \tag{A43}$$

where $p(x, t)$ and $S(x, t)$ are the solutions for the FPK equation and the Bellman–Hamilton–Jacobi equation, respectively. The equation:

$$\frac{\partial S(x,t)}{\partial t}=L_0S(x,t)-\nu_0c(x)-\nu\left(\frac{\partial S(x,t)}{\partial t}\right)^{\mathrm{T}}\times N(x)\left(\frac{\partial S(x,t)}{\partial t}\right) \tag{A44}$$

where ν_0 and ν are positive constants and $N(x)$ is an $(n\times n)$ positive-definite symmetric matrix, and may be viewed formally as the (time-reversal) Beliman–Hamilton–Jacobi equation for the control process:

$$\mathrm{d}\xi(\tau)=b(\xi(\tau))\,\mathrm{d}\tau+u(\tau)\,\mathrm{d}\tau+G(\xi(\tau))\,\mathrm{d}w(\tau),\ 0\le\tau\le t \tag{A45}$$

with the feedback control $u(\tau)=u(\tau,\xi(\tau))$ and the cost functional

$$J(u)=M_{x_0}\{S_0[\xi(\tau)]+\int_0^t L[\tau,\xi(\tau),u(\tau)]\,\mathrm{d}\tau\} \tag{A46}$$

with $L(\tau,x,u)=1/4u^{\mathrm{T}}N^{-1}(x)u-\nu_0c(x)$ and $b(x)=[b_1(x),\ldots,b_n(x)]^{\mathrm{T}}$. The FPK equation:

$$\frac{\partial p(t,x)}{\partial t}=(L_0+c(x))p(t,x) \tag{A47}$$

with $p(0,x)=p_0(x)$ defined for diffusion process that is described by n-vector Itô stochastic differential equation:

$$\mathrm{d}x(t)=f(x(t))\,\mathrm{d}t+G(x(t))\,\mathrm{d}w(t),\ 0\le t\le T \tag{A48}$$

Indeed, the Cole–Hopf transformation $v(t,x)=-\ln p_0(t,x)$ change [eqn (A47)] into a (time-reversal) dynamic programming equation for $v(t,x)$ of the form eqn (A44) with $\nu_0=1$, $\nu=1/2$ and $N(x)=G(x)G^T(x)$. If $p(t,x)$ is the solution of the FPK eqn (A47) and $S(t,x)$ is the solution of the Bellman–Hamilton–Jacobi eqn (A44), then the function $\psi(t,x)$ defined by eqn (A43) satisfies the nonlinear Schrödinger-like equation:

$$\frac{\partial\psi(t,x)}{\partial t}=[L_0+U(x;\psi)]\psi(t,x) \tag{A49}$$

with initial conditions, $\psi(0,x)=\sqrt{p_0(x)}\exp\{iS_0(x)\}$ and

$$U(x;\psi)=\frac{1}{2}(1-2i\nu_0)c(x)+\frac{1}{2}\left(\frac{\partial}{\partial x}\ln\psi^*(t,x)\right)G(x)G^{\mathrm{T}}(x)\times\left(\frac{\partial}{\partial x}\ln\psi^*(t,x)\right)-\frac{1}{4}\left(\frac{\partial}{\partial x}\ln\frac{\psi^*(t,x)}{\psi(t,x)}\right)^{\mathrm{T}}\times\{G(x)G^{\mathrm{T}}(x)-i\nu N(x)\}\left(\frac{\partial}{\partial x}\ln\frac{\psi^*(t,x)}{\psi(t,x)}\right) \tag{A49a}$$

where $\psi^*(t,x)$ is the complex conjugate of $\psi(t,x)$ and $S(t,x)=1/2i\ln(\psi^*(t,x))/(\psi(t,x))$.

If the function $\pi(x)$ is the nontrivial solution to $\partial\pi(x)/\partial x+b_0(x)\pi(x)=0$ with $b_0(x)=\{G(x)G^{\mathrm{T}}(x)\}^{-1}B(x)$ [$B(x)$ is the diffusion coefficient in the FPK eqn (A47)] then the transformation $\psi_0(t,x)=1/(\pi(x))\psi(t,x)$ satisfies:

$$\frac{\partial\psi_0(t,x)}{\partial t}=\left[\frac{1}{2}\mathrm{tr}\left\{G^{\mathrm{T}}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right)^{\mathrm{T}}G(x)\right\}+U_0(x;\psi_0)\right]\times\psi_0(t,x) \tag{A50}$$

with

$$U_0(t,x)=\frac{1}{2}\left(\frac{\partial}{\partial x}\ln\psi^*\right)^{\mathrm{T}}G(x)G^{\mathrm{T}}(x)\left(\frac{\partial}{\partial x}\ln\psi_0^*\right)-\frac{1}{4}\left(\frac{\partial}{\partial x}\ln\frac{\psi_0^*}{\psi_0}\right)^{\mathrm{T}}\{G(x)G^{\mathrm{T}}(x)-i\nu N(x)\}\times\left(\frac{\partial}{\partial x}\ln\frac{\psi_0^*}{\psi_0}\right) \tag{A50a}$$

For the scalar case eqn (A50) reduces to:

$$\frac{\partial\psi_0(t,x)}{\partial t}=a(x)\frac{\partial^2\psi_0(t,x)}{\partial x^2}+U_0(x;\psi_0)\psi_0(t,x) \tag{A51}$$

which is just the nonlinear Schrödinger equation if the time is formally replaced by the imaginary time $t/i\hbar$. A type of this nonlinear Schrödinger equation with a complex random nonlinear potential is known in quantum physics as the Schrödinger–Langevian equation as in the case of the Doebner-Golin equation,[69,70] where the potential $U_0(x;\psi)$ has the form $(i\alpha\Delta\psi/\psi+i\beta|\nabla\psi|^2/|\psi|^2)$. If in the FPK eqn (A47), introduce the transformation $p_1(t,x)=\pi(x)^{-1}p(t,x)$, then the function $p_1(t,x)$ satisfies the linear differential equation:

$$\frac{\partial p_1(t,x)}{\partial t}=\left[\frac{1}{2}\mathrm{tr}\left\{G^{\mathrm{T}}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right)^{\mathrm{T}}G(x)\right\}+W(x)\right]P_1(t,x) \tag{A52}$$

where

$$W(x)=-\frac{1}{2}\mathrm{tr}\left\{G^{\mathrm{T}}\frac{\partial b_0^{\mathrm{T}}(x)}{\partial x}G(x)\right\}-\frac{1}{2}b_0^{\mathrm{T}}(x)G(x)G^{\mathrm{T}}(x)b_0+c(x) \tag{A52a}$$

If $G(x)\equiv constant=G$, then $W(x)$ is given by $W(x)=-1/2[\partial f(x)/\partial x+G^{-2}f^2(x)]$.

Let $f(x)$ satisfy the Riccati equation for given function $r(x)$:

$$\frac{\partial f(x)}{\partial x}+G^{-2}f^2(x)+r(x)=0 \tag{A52b}$$

then $W(x)$ becomes as $W(x)=r(x)/2$. Thus, eqn (A52) reduces to the imaginary time analogue of the Schr ödinger equation with the given potential $r(x)/2$, i.e.:

$$\frac{\partial p_1(t,x)}{\partial t}=a\frac{\partial^2p_1(t,x)}{\partial x^2}+\frac{1}{2}r(x)p_1(t,x),\ a=\frac{1}{2}G^2. \tag{A53}$$

For $f(x)\equiv 0$, eqn (A52) reduces to the familiar heat equation. If $q(t,x)=-2\ln\partial/\partial xp_1(t,x)$, then this transforms eqn (A53) into the one-dimensional Navier–Stokes equation.

$$\frac{\partial q(t,x)}{\partial t}+aq(t,x)\frac{\partial q(t,x)}{\partial x}=-\frac{\partial r(x)}{\partial x}+a\frac{\partial^2 q(t,x)}{\partial x^2} \quad \text{(A54)}$$

In the case where the potential $r(x)$ is zero, eqn (A54) reduces to Bürger's equation.

Consider the following nonlinear FPK equation:[71]

$$\frac{\partial p}{\partial t}+\nabla j=0;\ j=-B[2p(1+\kappa p)\nabla U+\nabla p] \quad \text{(A55)}$$

where $U(t,x)$ is a given arbitrary function, B is the diffusion coefficient, and $\kappa \in R$. The introduction of the factor $1+\kappa p$ has its origin in the presence of the exclusion–inclusion principle and allows us to take into account many particle quantum effects. The transition probability from the site x to x' is defined as $P(t,x \rightarrow x')\ \ x') = r(t,x,x')p(t,x)[1+\kappa p(t,x')]$ with $r(t,x,x')$ as the transition rate. If $\kappa > 0$, $P(t,x \rightarrow x')$ introduces an inclusion principle, and in the case where $\kappa < 0$, the $P(t,x \rightarrow x')$ takes into account the Pauli exlusion principle. In Nelson's stochastic mechanics, the forward and backward velocities $v^{(\pm)} = -2B\nabla U^{(\pm)}$ $(U^{(+)} = U)$ and the FPK equation will thus give us:

$$\frac{\partial p}{\partial t}=\nabla[vp(1+\kappa p)]=0,\ v=\frac{1}{2}[v^{(+)}+v^{(-)}] \quad \text{(A56)}$$

For $v=\nabla\xi$, we obtain $\nabla\xi = j/p(1+\kappa p)$ and the nonlinear Schrödinger equation with $B=\hbar/2m$ and $p=|\psi|^2$ described from the FPK eqn (A56) as:

$$i\hbar\frac{\partial \psi}{\partial t}=-\frac{\hbar^2}{2m}\Delta\psi+V\psi+[W(p)+iQ(p,j)]\psi \quad \text{(A57)}$$

with

$$W(p)=\kappa\frac{\hbar^2}{4m}\left[\frac{\Delta p}{1+\kappa p}+\frac{2-\kappa p}{2p}\frac{(\nabla p)^2}{(1+\kappa p)^2}\right]$$

$$Q(p,j)=-\kappa\frac{\hbar}{2p}\nabla\left(\frac{pj}{1+\kappa p}\right),\ V=E-m\frac{\partial\xi}{\partial t}-\frac{1}{2}m(\nabla\xi)+mB^2\left[\frac{\Delta p}{p}-\frac{1}{2}\left(\frac{\nabla p}{p}\right)^2\right]-\kappa mB^2\left[\frac{\Delta p}{1+\kappa p}+\frac{2-\kappa p}{2p}\frac{(\nabla p)^2}{(1+\kappa p)^2}\right] \quad \text{(57a)}$$

For the transformation eqn (A43) the eqns (A55) and (A57) can be written as:

$$\frac{\partial p}{\partial t}+\nabla\left[\frac{\Delta S}{m}p(1+\kappa p)\right]=0 \quad \text{(A58)}$$

$$\frac{\partial S}{\partial t}+\frac{(\nabla S)^2}{2m}+V+V_q(p)+W(p)=0 \quad \text{(A59)}$$

where $V_q(p) = -(\hbar^2/2m)\ \Delta p^{1/2}/p^{1/2}$ is the Madelung–Bohm quantum potential. Eqn (A59) is a generalized quantum Hamilton–Jacobi equation and can be obtained in the frame of a classical stochastic process with $B = \hbar/2m$.

Example A5. The FPK equation as in the form [eqn (A3)]:

$$\frac{\partial w}{\partial t}=\frac{\partial^2 w}{\partial x^2}-2\frac{\partial f(x)w}{\partial x} \quad \text{(A60)}$$

describes the diffusion in an external potential

$$U(x)\equiv -2\int f(x)\,\mathrm{d}x \quad \text{(A61)}$$

If to define $\Psi(x,t) \equiv e^{U(x)/2}f(x)$ then eqn (A60) is modified to the Schrödinger equation:[72]

$$\frac{\partial \Psi}{\partial t}=\frac{\partial^2 \Psi}{\partial x^2}-(f'+f^2)\Psi \quad \text{(A62)}$$

On other hand, if in the diffusion equation with the space-dependent diffusion coefficient

$$\frac{\partial \xi}{\partial t}=\frac{\partial}{\partial x}\left(B(x)\frac{\partial \xi}{\partial x}\right) \quad \text{(A63)}$$

changes the scale of the space as

$$x\rightarrow X\equiv\int_0^x\frac{\mathrm{d}x}{\sqrt{B(x)}} \quad \text{(A64)}$$

then eqn (A63) becomes

$$\frac{\partial \xi}{\partial t}=\frac{\partial^2 \xi}{\partial X^2}+\left(\frac{\mathrm{d}}{\mathrm{d}X}\ln\sqrt{B(x)}\right)\frac{\partial \xi}{\partial X} \quad \text{(A65)}$$

In addition, eqn (A60) is transformed into:

$$\frac{\partial \eta}{\partial t}=\frac{\partial^2 \eta}{\partial x^2}+2f\frac{\partial \eta}{\partial x} \quad \text{(A66)}$$

where

$$\eta(x,t)=\mathrm{e}^{U(x)}w(x,t) \quad \text{(A66a)}$$

From eqns (A66) and (A66a), we find that eqns (A60) and (A63) are equivalent. Moreover, the square root of the diffusion coefficient in eqn (A65) corresponds to the stationary solution $\exp[-U(x)]$ for the FPK eqn (A60). The more general form [eqn (A3)] also reduces to eqn (A63) or eqn (A66) if the scale of x and ξ is adjusted appropriately. A connection between the Schrödinger equation and the FPK equation was studied also in the referenced paper.[73]