ELSEVIER

PII: S0266-8920(97)00024-6

Prob. Engng. Mech. Vol. 13, No. 3, pp. 205–226, 1998

0266-8920/98 $19.00+0.00

Stochastic analysis of time-variant nonlinear dynamic systems. Part 2: methods of statistical moments, statistical linearization and the FPK equation

S. V. Ulyanov[a,*], M. Feng[b], V. S. Ulyanov[a] & K. Yamafuji[a]

[a]*Department of Mechanical and Control Engineering, The University of Electro-Communications, Chofu, Tokyo 182, Japan*
[b]*Department of Civil and Environmental Engineering, Engineering Gateway E4139, University of California, Irvine, CA 92717, USA*

In this, the second of a two part paper, the applications of the Fokker–Planck–Kolmogorov (FPK) method to stochastic analysis of time-variant nonlinear systems are considered. A new class of dynamic systems with stochastic nonlinearity and jump parametric exitations is introduced. The comparison of accuracy of different statistical methods such as statistical linearization is discussed. © 1998 Elsevier Science Limited.

1 INTRODUCTION

In Part 1 of this paper,[1] a stochastic analysis of time-invariant nonlinear systems subjected to random linear and nonlinear parametric excitations was conducted on the basis of the FPK equation. In this paper (Part 2), two types of time-variant nonlinear dynamic systems are considered: (1) dynamic systems with stochastic nonlinearity; and (2) dynamic systems with random jump parameters or a sliding mode in present of noise. Random change of charactistics is considered as: (1) a stochastic event that does not depend on phase coordinates of the system; and (2) a function of phase coordinates of the system. The time-invariant dynamic system is obtained as a special case in which the change of characteristics has a zero probability for one random realization of motion. These two types of dynamic systems can be united on the basis of a common mathematical tool for statistical analysis of parametric systems.[1]

For engineering probabilistic analysis of complex nonlinear systems, different mixed (hybrid) methods are used.[2–11] In Part 1, the exact solution of the FPK equation is obtained on the basis of asymptotic analysis of the nonlinear dynamic behavior of mechanical system with parametric excitations. In this paper, applications of the FPK method to stochastic analysis of time-variant nonlinear systems are considered. A new class of dynamic systems with stochastic nonlinearity and jump parametric excitations is introduced. A comparison between the method of statistical linearization[5] and method of statistical moments[4,12] are performed.

*Corresponding author. Tel: 0081 424 832161x3734; Fax: 0081 424 843327; e-mail: sergei@yama.mce.ueu.ac.jp

2 STATISTICAL ANALYSIS ON THE BASIS OF STATISTICAL LINEARIZATION

Consider a nonlinear dynamic system as:

$$\ddot{x} + 2\beta_0\dot{x} + F(x,\dot{x},\ddot{x}) = \nu\eta(t) \tag{1}$$

where

$$F(x,\dot{x},\ddot{x}) = \Omega^2 x + \varepsilon_0 x^2\dot{x} + \gamma_0 x^3 + 2\kappa x(x\ddot{x} + \dot{x}^2) \tag{2}$$

According to the method of statistical linearization, the nonlinear function in eqn (2) can be presented as:

$$\begin{aligned} F(x,\dot{x},\ddot{x}) = {} & F_0(m_x, \dot{m}_x, \ddot{m}_x, \sigma_x^2, \sigma_{\dot{x}}^2, \sigma_{\ddot{x}}^2, R_{x\ddot{x}}) \\ & + K_{1x}(m_x,\ldots,R_{x\ddot{x}})x^0 + K_{1\dot{x}}(m_x,\ldots,R_{x\ddot{x}})\dot{x}^0 \\ & + K_{1\ddot{x}}(m_x,\ldots,R_{x\ddot{x}})\ddot{x}^0 \end{aligned} \tag{3}$$

where

$$\begin{aligned} F_0(m_x,\ldots,R_{x\ddot{x}}) = {} & \Omega^2 m_x + \gamma_0(m_x^3 + 3\sigma_x^2 m_x) \\ & + \varepsilon_0(\dot{m}_x m_x^2 + \dot{m}_x\sigma_x^2) \\ & + 2\kappa(m_x^2\ddot{m}_x + \sigma_x^2\ddot{m}_x + 2m_x R_{x\ddot{x}} + m_x\dot{m}_x^2 \\ & + m_x\sigma_{\dot{x}}^2) \end{aligned}$$

$$K_{1x} = \frac{\partial F_0}{\partial m_x} = \Omega^2 + 3\gamma_0(m_x^2 + \sigma_x^2) + 2\varepsilon_0 \dot{m}_x m_x$$

$$+ 2\kappa(2m_x\ddot{m}_x + 2R_{x\ddot{x}} + \dot{m}_x^2 + \sigma_{\dot{x}}^2);$$

$$K_{1\dot{x}} = \frac{\partial F_0}{\partial \dot{m}_x} = \varepsilon_0(m_x^2 + \sigma_x^2) + 4\kappa\dot{m}_x m_x;$$

$$K_{1\ddot{x}} = \frac{\partial F_0}{\partial \ddot{m}_x} = 2\kappa(m_x^2 + \sigma_x^2);$$

$$R_{x\ddot{x}} = -\sigma_{\dot{x}}^2$$

Suppose that

$$x = mx + x_0$$

where x^0 is a centered stochastic process with zero mean. Substituting eqn (4) into eqn (1), one obtains:

$$\ddot{m}_x + 2\beta_0\dot{m}_x + F_0(m_x, \dot{m}_x, \ddot{m}_x, \sigma_x^2, \sigma_{\dot{x}}^2, \sigma_{\ddot{x}}^2, R_{x\ddot{x}}) = \nu m_\eta(t) \tag{5}$$

and

$$\ddot{x}^0 + 2\beta_0\dot{x}^0 + k_{1x}x^0 + k_{1\dot{x}}\dot{x}^0 + k_{1\ddot{x}}\ddot{x}^0 = \nu\eta^0 \tag{6}$$

In a stationary regime where, $m_\eta(t)$ = *constant* and $\dot{m}_x = \ddot{m}_x = 0$, eqns (5) and (6) become:

$$\Omega^2 m_x + \gamma_0(m_x^3 + 3\sigma_x^2 m_x) - 2\kappa m_x\sigma_{\dot{x}}^2 = \nu m_\eta \tag{7}$$

$$\ddot{x}^0 + \frac{2\beta_0 + k_{1\dot{x}}}{1 + k_{1\ddot{x}}}\dot{x}^0 + \frac{k_{1x}}{1 + k_{1\ddot{x}}}x^0 = \frac{\nu}{1 + k_{1\ddot{x}}}\eta^0 \tag{8}$$

where

$$k_{1x} = \Omega^2 + 3\gamma_0(m_x^2 + \sigma_x^2) - 2\kappa\sigma_{\dot{x}}^2;\ k_{1\dot{x}} = \varepsilon_0(m_x^2 + \sigma_x^2) \tag{8a}$$

Example 1. A stationary random process $\eta(t)$ has a power spectral density $S_\eta(\omega) = (\sigma_\eta^2)/(\pi)\ (\beta)/((\beta^2 + \omega^2))$, where the parameters σ_η^2, β are of arbitrary real constants. From eqn (8), one obtains:

$$\sigma_x^2 = \frac{\sigma_\eta^2}{k_{1x}} \frac{[\beta(1 + k_{1\ddot{x}}) + (2\beta_0 + k_{1\dot{x}})]}{\{k_{1x}[\beta k_{1\ddot{x}} + (2\beta_0 + k_{1\dot{x}})] + \beta(2\beta_0 + k_{1\dot{x}})[\beta(1 + k_{1\ddot{x}}) + (2\beta_0 + k_{1\dot{x}})]\}}$$

$$\sigma_{\dot{x}}^2 = \sigma_x^2 - \frac{\sigma_\eta^2}{k_{1x}} \frac{\beta k_{1x}}{\{k_{1x}[\beta k_{1\ddot{x}} + (2\beta_0 + k_{1\dot{x}})] + \beta(\beta_0 + k_{1\dot{x}})[\beta(1 + k_{1\ddot{x}}) + (2\beta_0 + k_{1\dot{x}})]\}} \tag{9}$$

Therefore, the following nonlinear algebraic equations are obtained for calculating m_x, σ_x^2 and $\sigma_{\dot{x}}^2$:

$$\Omega^2 m_x + \gamma_0(m_x^3 + 3\sigma_x^2 m_x) - 2\kappa m_x\sigma_{\dot{x}}^2 = \nu m_\eta \tag{9a}$$

$$\sigma_x^2 = \frac{\sigma_\eta^2}{k_{1x}} \frac{[\beta(1 + k_{1\ddot{x}}) + (2\beta_0 + k_{1\dot{x}})]}{\{k_{1x}[\beta k_{1\ddot{x}} + (2\beta_0 + k_{1\dot{x}})] + \beta(2\beta_0 + k_{1\dot{x}})[\beta(1 + k_{1\ddot{x}}) + (2\beta_0 + k_{1\dot{x}})]\}} \tag{9b}$$

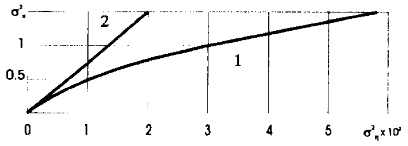

Fig. 1. Graphs of relationship $\sigma_x^2 = f(\sigma_\eta^2)$.

$$\sigma_{\dot{x}}^2 = \sigma_x^2 - \frac{\sigma_\eta^2}{k_{1x}} \frac{\beta k_{1x}}{\{k_{1x}[\beta k_{1\ddot{x}} + (2\beta_0 + k_{1\dot{x}})] + \beta(\beta_0 + k_{1\dot{x}})[\beta(1 + k_{1\ddot{x}}) + (2\beta_0 + k_{1\dot{x}})]\}} \tag{9c}$$

$$k_{1x} = \Omega^2 + 3\gamma_0(m_x^2 + \sigma_x^2) - 2\kappa\sigma_{\dot{x}}^2;\ k_{1\dot{x}} = \varepsilon_0(m_x^2 + \sigma_x^2) \tag{9d}$$

Eqns (9a), (9b), (9c) and (9d) are obtained using the successive substitution and graphic methods. In case $m_\eta(t) = 0$, one finds $m_x = 0$ and σ_x^2, and $\sigma_{\dot{x}}^2$ are obtained from eqns (9b) and (9c). The coefficients k_{1x}, $k_{1\dot{x}}$ and $k_{1\ddot{x}}$ become:

$$k_{1x} = \Omega^2 + 3\gamma_0\sigma_x^2 - 2\kappa\sigma_{\dot{x}}^2,\ k_{1\dot{x}} = \varepsilon_0\sigma_x^2,\ k_{1\ddot{x}} = 2\kappa\sigma_x^2 \tag{10}$$

Fig. 1 shows σ_x^2 as a function of σ_η^2 based on eqns (9a), (9b), (9c), (9d) and (10). Curve 1 represents a nonlinear system with parameters $\beta_0 = 0.2$; $\Omega^2 = 25$; $\varepsilon_0 = 0.5$; $\gamma_0 = 0.2$; $\kappa = 0$; and $\beta = 10$. Curve 2 represents a linear system with $\varepsilon_0 = \gamma_0 = \kappa = 0$. The presence of nonlinearity in eqns (9a), (9b), (9c) and (9d) restricts the increase of the vibration amplitude under external excitations.

3 METHODS OF STATISTICAL MOMENTS AND FPK EQUATION

Consider dynamic system described as

$$\dot{x} = F(x, t) + F_1(x, t)\eta \tag{11}$$

where x is output, $\eta(t)$ is a white noise with zero expectation, and F and F_1 are the nonlinear functions. Denote M as the symbol of expectation and x^m as the mth moment of x. Since:

$$\frac{d}{dt}M[x^m] = M\left[\frac{d}{dt}x^m\right] = mM\left[x^{m-1}\frac{dx}{dt}\right] \tag{12}$$

Substituting into eqn (12) the value of dx/dt from eqn (11), one obtains:

$$\frac{d}{dt}M[x^m] = mM[x^{m-1}F(x, t)] + mM[x^{m-1}F_1(x, t)\eta] \tag{13}$$

Remark 1. This pertains to a logical connection between the method of statistical moments and the FPK equations. Consider a series of infinitely short rectangular pulses along the time axis, statistically independent of each other. Assuming that the increment of the stochastic output process $\Delta x(t)$ caused by the effect of an instantaneous 'pulse' of white noise $\eta(t)$ is small. In this case, the second term of the right-hand side of eqn (13) becomes:

$$mM[x^{m-1}F_1(x,t)\eta] = mM\left[(x_0^{m-1} + (m-1)x_0^{m-2}\Delta x) \times \left\{F_1(x_0,t) + \frac{\partial F_1(x_0,t)}{\partial x_0}\Delta x\right\}\eta\right]$$
$$= mM\left[x_0^{m-1}F_1(x_0,t)\eta + x_0^{m-1}\frac{\partial F_1(x_0,t)}{\partial x_0}\Delta x\eta + (m-1)x_0^{m-2}F_1(x_0,t)\Delta x\eta + (m-1)x_0^{m-2}\frac{\partial F_1(x_0,t)}{\partial x_0} \times (\Delta x)^2\eta\right] \quad (14)$$

where x_0 is a component of the output process independent of $x(t)$ which is caused by proceeding 'pulses' of white noise and $x = x_0 + \Delta x(t)$. Since $\Delta x(t)$ is small, $(\Delta x)^2$ may be neglected. Thus:

$$mM[x^{m-1}F_1(x,t)\eta] = M\left[mx_0^{m-1}\frac{\partial F_1(x_0,t)}{\partial x_0} + m(m-1)x_0^{m-2}F_1(x_0,t)\right] \times M[\Delta x\eta] \quad (15)$$

The value of $M[\Delta x\eta]$ can be easily found from eqn (11). From eqn (11) one obtains:

$$\Delta x = \int_t^{t+\Delta t} F(x,\tau)\,d\tau + \int_t^{t+\Delta t} F_1(x,\tau)\eta(\tau)\,d\tau \quad (16)$$

After multiplying both sides of eqn (16) by $\eta(t)$ and averaging over the set for $\Delta t \to 0$, one obtains:

$$M[\Delta x\eta] = \frac{\sigma_\eta^2}{2}F_1(x,t) \quad (17)$$

where σ_η^2 is the spectral intensity of the white noise. Then eqn (13) takes the form:

$$\frac{d}{dt}M[x^m] = M\left[mx^{m-1}\left(F(x,t) + \frac{\sigma_\eta^2}{2}F_1(x,t)\frac{\partial F_1(x,t)}{\partial x}\right)\right] + M\left[m(m-1)\frac{\sigma_\eta^2}{2}x^{m-2}F_1^2(x,t)\right] \quad (18)$$

The equations derived can be used for an approximate calculation of the probability density function by representing it with an approximating series. The unknown coefficients can be found by substituting an appropriate series into eqn (18).[12]

Substituting into eqn (18)

$$\frac{d}{dt}M[x^m] = 0, F_1(x,t) = 1, F(x,t) = -F(x,t)$$

one obtains

$$M\left[x^{m-2}\left(xF(x) - \frac{\sigma_\eta^2}{2}(m-1)\right)\right] = 0 \quad (19)$$

It is obvious that if the positive values of $xF(x)$ are finite, there always exists some value m which makes eqn (19) meaningless. When $F(z)$ is approximated by a sign-changing power series, eqn (19) may have a solution for any m as long as the last term of the series is positive and contains z to an odd power.

On the other hand:

$$\frac{d}{dt}M[x^m] = \frac{d}{dt}\int_{-\infty}^{\infty} x^m p(x,t)\,dx = \int_{-\infty}^{\infty} x^m \frac{\partial p(x,t)}{\partial t}\,dx \quad (20)$$

where $p(x, t)$ is the probability density function of process $x(t)$ as in part 1. For an arbitrary and differentiable function $\Phi(x, t)$, the following equation is given:[12]

$$\int_{-\infty}^{\infty} x^{m-k}\Phi(x,t)p(x,t)\,dx = \frac{(-1)^k}{m(m-1)\ldots(m-k)}\int_{-\infty}^{\infty} x^m \frac{\partial^k}{\partial x^k}[\Phi(x,t)p(x,t)]\,dx \quad (21)$$

By integrating the left side of eqn (21) by parts and considering

$$\lim_{|x|\to 0} x^{m-i}\frac{\partial^i}{\partial x^i}[\Phi(x,t)p(x,t)] \to 0 \quad (21a)$$

we obtain

$$\int_{-\infty}^{\infty} x^m\left\{\frac{\partial p(x,t)}{\partial t} + \frac{\partial}{\partial x}\left[(F(x,t) + \frac{\sigma_\eta^2}{2}F_1(x,t)\frac{\partial F_1(x,t)}{\partial x})p(x,t)\right] - \frac{\sigma_\eta^2}{2}\frac{\partial^2}{\partial x^2}[F_1^2(x,t)p(x,t)]\right\}dx = 0 \quad (22)$$

eqn (22) should be true for any possible value of m only if

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left\{\left[\left(F(x,t) + \frac{\sigma_\eta^2}{2}F_1(x,t)\frac{\partial F_1(x,t)}{\partial x}\right)\right] \times p(x,t)\right\} + \frac{\sigma_\eta^2}{2}\frac{\partial^2}{\partial x^2}[F_1^2(x,t)p(x,t)] \quad (23)$$

eqn (23) is the FPK equation in which the coefficients of drift and diffusion are expressed directly through the characteristics of the nonlinear dynamic system in eqn (11) with the parametric random exctiations, (see Part 1 of this paper[1]).

Special Case. For $\partial F_1/\partial x = 0$, eqn (23) becomes

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}[F(x,t)p(x,t)] + \frac{\sigma_\eta^2}{2}\frac{\partial^2}{\partial x^2}[F_1^2(x,t)p(x,t)] \quad (24)$$

In this case, the wide band disturbances acting on the dynamic system of eqn (11) do not result in any variation in $F(x)$ that are correlated with them.

Example 2. Consider a dynamic system described by the stochastic differential equation:

$$\ddot{x}+\gamma\dot{x}+F(x)=\eta(t) \tag{25}$$

For simplicity, we assume $F(x, t) = -F(x, t)$ and introducing the symbols: $x = x_1$, $x_2 = \dot{x}_1$. Eqn (25) is rewritten as:

$$\dot{x}_1=x_2,\ \dot{x}_2=\eta(t)-\gamma x_2-F(x_1) \tag{26}$$

The rate of variation of the product moment $M[x_1^m x_2^n]$is:

$$\frac{d}{dt}M[x_1^m x_2^n]=mM\left[x_1^{m-1}x_2^n\frac{dx_1}{dt}\right]+nM\left[x_1^m x_2^{n-1}\frac{dx_2}{dt}\right] \tag{27}$$

Substituting into eqn (27) the value of the derivatives from eqn (26), one obtains:

$$\frac{d}{dt}M[x_1^m x_2^n]=mM[x_1^{m-1}x_2^{n+1}] - \gamma nM[x_1^m x_2^n]-nM[x_1^m x_2^{n-1}F(x_1)] + nM[x_1^m x_2^{n-1}\eta] \tag{28}$$

Write the last term in eqn (28) in the form:

$$nM[x_1^m x_2^{n-1}\eta]=nM[x_1^m x_{02}^{n-1}\eta] + n(n-1)M[x_1^m x_{02}^{n-2}]M[\Delta x_2\eta] \tag{29}$$

where x_{02} (as the component of x_2 not correlated with and Δx_2) is the increment caused by the effect of an instantaneous 'pulse' of white noise. Since:

$$M[\Delta x_2\eta]=\frac{\sigma_\eta^2}{2} \tag{30}$$

one obtains

$$\frac{d}{dt}M[x_1^m x_2^n]=mM[x_1^{m-1}x_2^{n+1}]-\gamma nM[x_1^m x_2^n] - nM[x_1^m x_2^{n-1}F(x_1)] + n(n-1)\frac{\sigma_\eta^2}{2}M[x_1^m x_2^{n-2}] \tag{31}$$

For the stationary mode (if it exists),

$$\frac{d}{dt}M[x_1^m x_2^n]=0 \tag{31a}$$

Assume $n = 0$, then from eqn (31) one obtains:

$$M[x_1^{m-1}x_2^n]=0 \tag{32}$$

i.e. x_1 and x_2 are independent at concurrent moment of time. If $m = 0$, then:

$$\gamma M[x_2^n]=(n-1)\frac{\sigma_\eta^2}{2}M[x_2^{n-1}] \tag{33}$$

This the moment of x_2 are linearly interrelated. When $n = 2$

$$M[x_2^2]=\sigma_{x_2}^2=\frac{\sigma_\eta^2}{2\gamma} \tag{34}$$

From eqns (33) and (34) one obtains:

$$M[x_2^{2m}]=\sigma_{x_2}^{2m}=(2m-1)\sigma_{x_2}^2\sigma_{x_2}^{2m-1} \tag{35}$$

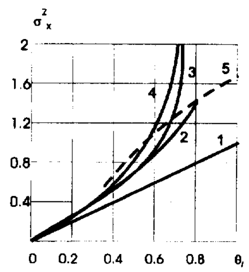

Fig. 2. Graphs of relationship $\sigma_x^2=f(\sigma_\eta^2=\theta_1)$.

eqn (35) can be used to calculate the moments of x_2 of any order.

Considering that x_1 and x_2 are independent at concurrent moments of time and assuming that $m = n = 2l$-1 and $d/dtM[(x_1x_2)^{2l-1}]=0$ in eqn (31) and the odd moments of x_1 and x_2 are equal to zero, one obtains:

$$\sigma_{x_1}^{2(m-1)}\sigma_{x_2}^{2m}-\sigma_{x_2}^{2(m-1)}M[\sigma_{x_1}^{2m-1}F(x_1)]=0 \tag{36}$$

Using eqns (33), (34) and (19), one obtains:

$$M[\sigma_{x_1}^{2m-1}F(x_1)]=2(m-1)\frac{\sigma_\eta^2}{2\sigma_{x_1}^2}\sigma_{x_1}^{2(m-1)} \tag{37}$$

When function $F(x)$ decreases more rapidly for $|x|\rightarrow 0$, the stationary solution of the moment equation devergents.[12]

Example 3: System Linearization. For dynamic system in eqn (25), assume:

$$F(x)=\sin x\cong x-\frac{x^3}{3!}+\frac{x^5}{5!} \tag{38}$$

eqn (37) for this case becomes

$$\sigma_{x_1}^{2m}-\frac{1}{3!}\sigma_{x_1}^{2(m+1)}+\frac{1}{5!}\sigma_{x_1}^{2(m+2)}=(2m-1)\sigma_l^2\sigma_{x_1}^{2(m-1)},$$

$$\sigma_l^2=\frac{\sigma_\eta^2}{2\gamma} \tag{31}$$

Neglecting $M[x_1^8]$ and $M[x_1^{10}]$ and $M[x_1^{12}]$ in eqn (37), one obtains:

$$\sigma_{x_1}^2=\sigma_l^2\frac{1-\frac{5}{6}\sigma_l^2}{1-\frac{4}{3}\sigma_l^2+\frac{1}{8}\sigma_l^4} \tag{40}$$

where σ_l^2 is the output dispersion of the corresponding linear system.

Expanding eqn (40) in a series in powers of σ_l^2 gives:

$$\sigma_{x_1}^2=\sigma_l^2+\frac{1}{2}(\sigma_l^2)^2+\frac{13}{24}(\sigma_l^2)^3+\ldots \tag{41}$$

eqn (41) is the same as the solution obtained using the method of functional series.[2,12]

Fig. 2 shows graphs of the relationship $\sigma_{x_1}^2=f(\sigma_l^2)$, constructed, respectively, according to the formula of linear theory (curve 1), the method of functional series (curve 2), statistical linearization (curve 3), eqn (40) (curve 4), and the FPK equation from Ref.[1] (curve 5). A comparison of these curves indicates that for a system without parametric excitations all the methods [2,12] give approximately equivalent results over quite a wide range of variation of σ_l^2.

4 STATISTICAL ANALYSIS OF DYNAMIC SYSTEMS WITH STOCHASTIC CORRELATED NONLINEARITITES AND RANDOM JUMP PARAMETRIC EXCITATIONS OF STRUCTURE

As mentioned in Section 1, two types of time-variant nonlinear dynamic systems are considered: (1) dynamic systems with stochastic correlated nonlinearities; and (2) dynamic systems with random jump parameters or a sliding random noise mode. A random change in characteristics is considered as: (1) a stochastic event that does not depend on the phase coordinates of the system; and (2) a function of the phase coordinates of the dynamic system. The time-invariant dynamic system described by eqn (11) or eqn (25) is obtained as a special case in which the change in characteristics has zero probability for one random realization of motion. These two types of dynamic systems can be united on the basis of a common mathematical tool for statistical analysis of parametric systems.[1]

Applications of the FPK method[1] to stochastic analysis of these time-variant nonlinear systems are considered.

4.1 Case 1: dynamic systems with stochastic correlated nonlinearities and parametric excitations

Consider a class of dynamic systems described by the stochastic differential equations:

$$\frac{\mathrm{d}x_k}{\mathrm{d}t} = \sum_{l=1}^{n} [a_{kl}(t) + z_{kl}(t)]F_{kl}(x, \boldsymbol{\alpha}(t)) + N_k(t) \tag{42}$$

where $a_{kl}(t)$ is a function of the time; $F_{kl}(x, \boldsymbol{\alpha}(t))$ is a nonlinear function of vector variables $x(t)$ and $\alpha(t)$; $\boldsymbol{\alpha}(t)$ is an n-dimensional random vector process describing stochastic variation of parameters in the nonlinear function; $z_{kl}(t)$ is a random parametric excitation; and $Nk(t)$ is a random excitation. The random functions $z_{kl}(t)$, $\boldsymbol{\alpha}(t)$ and $N_k(t)$ are δ-correlated Gaussian processes with means $\langle z_{kl}(t)\rangle$, $\langle\boldsymbol{\alpha}(t)\rangle$, $\langle N_k(t)\rangle$ and intensity coefficients $G^z_{klp}(t)$, $G^{\alpha}(t)$, $G^N_k(t)$, $G^{z\alpha}_{klpq}(t)$, $G^{zN}_{klp}(t)$, $G^{\alpha N}_{klp}(t)$. A generalized case is considered where stochastic processes $z_{kl}(t)$, $\alpha(t)$ and $N_k(t)$ are random correlated processes.

In accordance with the method of stochastic linearization (see Example 1), the stochastic nonlinearity in $F_{kl}(x,\boldsymbol{\alpha}(t))$ is defined as statistically equivalent series:[2]

$$F_{kl}(x, a) = F_{0kl} + \sum_{q=1}^{n} K^{(1)}_{klq}x^0_q + \sum_{q=1}^{n} K^{(2)}_{klq}\alpha^0_q \tag{43}$$

where $K^{(1)}$ and $K^{(2)}$ are statistically equivalent gains on centered components x^0_q and α^0, respectively. The calculation of gains $K^{(1)}$ and $K^{(2)}$ is given in Ref.[2].

Define:

$$K^{(1)}_{kl}(x) = F_{0kl} + \sum_{q=1}^{n} K^{(1)}_{klq}x^0_q \tag{44}$$

The coefficients of drift and diffusion, $A_k(t, x)$ and $B_{kl}(t, x)$, in the FPK equation are defined as:[1]

$$A_k(t,x) = \lim_{\Delta x \to 0} \frac{M[\Delta x_k | X = x]}{\Delta t};$$

$$B_{kl}(t,x) = \lim_{\Delta x \to 0} \frac{M[\Delta x_k \Delta x_l | X = x]}{\Delta t} \tag{45}$$

From eqns (42)–(44), one obtains:

$$\begin{aligned}\Delta x_k = \sum_{l=1}^{n} \int_t^{t+\Delta t} & [(a_{kl}(\tau) + \langle z_{kl}(t)\rangle) + z^0_{kl}]\{F^{(1)}_{kl}(x(t)) \\ & + \sum_{q=1}^{n} K^{(2)}_{klq}\alpha^0_q + \sum_{p,q=1}^{n} K^{(1)}_{klp} \int_t^{\tau} ((\alpha_{pq}(\xi) + \langle z_{pq}(\xi)\rangle) \\ & + z^0_{pq}(\xi))[F^{(1)}_{pq}(x(\xi)) + \sum_{r=1}^{n} K^{(2)}_{pqr}\alpha^0_r] + (\langle N_p(\xi)\rangle \\ & + N^0_p(\xi)))\,\mathrm{d}\xi\}\,\mathrm{d}\tau + \int_t^{t+\Delta t} (\langle N_k(\tau)\rangle + N^0_k(\tau))\,\mathrm{d}\tau; \\ t \le \tau \le t + \Delta &\end{aligned} \tag{46}$$

Calculating the conditional expectation in eqn (45) gives:

$$\begin{aligned}A_k(t,x) = & \sum_{l=1}^{n} [(a_{kl}(t) + \langle z_{kl}(t)\rangle)F^{(1)}_{kl}(x)] \\ & + \frac{1}{2}\sum_{l,p,q,r=1}^{n} K^{(1)}_{klp}K^{(2)}_{pqr}((a_{kl}(t) + \langle z_{kl}(t)\rangle)G^{z\alpha}_{pqr} \\ & + (a_{pq}(t) + \langle z_{pq}(t)\rangle)G^{z\alpha}_{klr}) + \sum_{l,q=1}^{n} K^{(1)}_{klq}G^{z\alpha}_{klq} \\ & + \frac{1}{2}\sum_{l,p,q=1}^{n} K^{(1)}_{klp}G^{zN}_{klp} + \frac{1}{2}\sum_{l,p,q=1}^{n} K^{(1)}_{klp}F^{(1)}_{pq}(x)G^{z}_{klpq} \\ & + \langle N_k(t)\rangle;\ (k = 1,\ 2, \ldots, n)\end{aligned} \tag{47}$$

Calculating the product $(\Delta x_k \Delta x_\nu)$ from eqn (46) and substituting it into eqn (45) results in:

$$\begin{aligned}2B_{k\nu}(x,t) = & \sum_{l,p,q,r=1}^{n} (a_{kl}(t) + \langle z_{kl}(t)\rangle)(a_{\nu p}(t) \\ & + \langle z_{\nu p}(t)\rangle)K^{(2)}_{klp}K^{(2)}_{\nu pr}G^{\alpha}_{qr} \\ & + \sum_{l,p,q=1}^{n} (a_{kl}(t) + \langle z_{kl}(t)\rangle)(K^{(2)}_{klq}F^{(1)}_{\nu p}(x)G^{z\alpha}_{q\nu p} \\ & + K^{(2)}_{\nu pq}F^{(1)}_{kl}(x)G^{z\alpha}_{\nu pq}) + \sum_{l,p,q=1}^{n} (a_{\nu p}(t) \\ & + \langle z_{\nu p}(t)\rangle)(K^{(2)}_{klq}F^{(1)}_{\nu p}(x)G^{z\alpha}_{klq} \\ & + K^{(2)}_{\nu pq}F^{(1)}_{kl}(x)G^{z\alpha}_{klq}) + \sum_{l,p=1}^{n} F^{(1)}_{kl}(x)F^{(1)}_{\nu p}(x)G^{z}_{kl\nu p}\end{aligned}$$

$$+\sum_{l,q=1}^{n}[(a_{kl}(t)+\langle z_{kl}(t)\rangle)K_{klq}^{(2)}G_{\nu q}^{\alpha N}$$
$$+(a_{\nu l}(t)+\langle z_{\nu l}(t)\rangle)K_{\nu lq}^{(2)}G_{kq}^{\alpha N}]$$
$$+\sum_{l=1}^{n}[F_{kl}^{(1)}(x)G_{kl\nu}^{zN}+F_{\nu l}^{(1)}(x)G_{\nu lk}^{zN}]+G_{k\nu}^{zN} \quad (48)$$

When $\alpha(t)\equiv 0$, eqns (47) and (48) provide the same results as those in Ref. [13]. From eqn (44) one obtains $\partial F_{ki}(x)/\partial x_p=K_{klp}^{(1)}$. Coefficients of drift, $A_k(t,x)$ and diffusion, $B_{k\nu}(t,x)$, defined in eqn (45), coincide with the results obtained from the symmetrization integrals as in Ref.[2].

Using the drift and diffusion coefficients of the FPK equation, the first statistical moments are defineded as:

$$\frac{d}{dt}M[x_k]=\frac{d}{dt}m_k=\langle A_k\rangle;\ \frac{d}{dt}M[x_kx_\nu]=\frac{d}{dt}\theta_{k\nu}$$
$$=\langle x_kA_\nu+x_\nu A_k+2B_{k\nu}\rangle \quad (49)$$

and mixed statistical moments as

$$\frac{d}{dt}M[x_1^{r_1}x_2^{r_2},\ldots,x_n^{r_n}]=\frac{d}{dt}\alpha_{r_1r_2,\ldots,r_n}$$
$$=\sum_{k=1}^{n}r_k\left\langle A_kx_k^{r_k-1}\prod_{p=1}^{n}x_p^{r_p}\right\rangle+\sum_{k=1}^{n}r_k(r_k-1)$$
$$\times\left\langle B_{kk}x_k^{r_k-2}\prod_{p=1}^{n}x_p^{r_p}\right\rangle$$
$$+\sum_{k,\nu=1;k\neq\nu}^{n}\left\langle B_{k\nu}x_k^{r_k-1}x_\nu^{r_\nu-1}\prod_{p=1}^{n}x_p^{r_p}\right\rangle \quad (50)$$

Example 4: Stochastic nonlinearity. Consider the dynamic system described as:

$$\ddot{x}+2\beta_0\dot{x}+\Omega^2x+\varepsilon_0F(x,\dot{x})=\eta(t), \quad (51)$$

where $\eta(t)$ is δ-correlated random process with zero mean and intensity σ_η^2; β_0, Ω^2 and ε_0 are constant parameters; $F(x,\dot{x})=1/3\ d/dt\varphi(t)$ and $\varphi(t)$ is a stochastic nonlinearity with conditional expectation and autocorrelation function as

$$m_\varphi=x^3;R_\varphi(\tau)=\sigma_\varphi^2s(\tau);\ s(\tau)=e^{-\alpha\tau};\ \sigma_\varphi^2=constant$$

When $\sigma_\varphi^2=0$, eqn (51) describes the system with nonlinear damping: $(1/3)\varepsilon_0\ d/dt\varphi(t)=\varepsilon_0x^2\dot{x}$.In accordance with the method of statistical linearization, the stochastic nonlinear function is defined as $\varphi_0(t)=\varphi_0+k_1x^0+\psi$. In the stationary regime $\varphi_0=0$ (as the conditional expection is a symmetric function and the input signal has a zero mean), $k_1=3\sigma_x^2$ and eqn (51) becomes:

$$\ddot{x}+(2\beta_0+\varepsilon_0\sigma_x^2)\dot{x}+\Omega^2x=\eta(t)-\frac{1}{3}\varepsilon_0\dot{\psi} \quad (52)$$

The variance of output motion is defined as:

$$\sigma_x^2=\int_{-\infty}^{\infty}\frac{S_\eta(\omega)+\frac{1}{9}\varepsilon_0|i\omega|^2S_\psi(\omega)}{|(i\omega)^2+(2\beta_0+\varepsilon_0\sigma_x^2)i\omega+\Omega^2|^2}d\omega \quad (53)$$

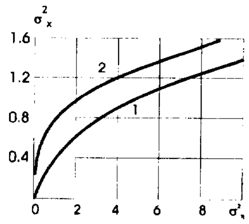

Fig. 3. Graphs of relationship $\sigma_x^2=f(\sigma_\psi^2,\sigma_\eta^2)$.

For $S_\psi=(\sigma_\psi^2\alpha)/(\pi(\omega^2+\alpha^2))$:

$$\sigma_x^2=\frac{\pi\sigma_\eta^2}{\Omega^2(2\beta_0+\varepsilon_0\sigma_x^2)}$$
$$-\frac{2\pi\sigma_\psi^2\alpha\varepsilon_0^2}{9(2\beta_0+\varepsilon_0\sigma_x^2)[\Omega^2+\alpha^2+\alpha(2\beta_0+\varepsilon_0\sigma_x^2)]}$$
$$=\frac{\{9\pi\sigma_\eta^2(\Omega^2+\alpha^2+2\alpha\beta_0)-2\pi\sigma_\psi^2\alpha\varepsilon_0^2\Omega^2\}+9\pi\sigma_\eta^2\sigma_x^2\varepsilon_0^2\alpha}{9\Omega^2(2\beta_0+\varepsilon_0\sigma_x^2)[\Omega^2+\alpha^2+\alpha(2\beta_0+\varepsilon_0\sigma_x^2)]} \quad (54)$$

For $\sigma_\psi^2=0$:

$$\sigma_x^2=\frac{\beta_0\Omega^2}{\varepsilon_0}[\sqrt{1+\theta}-1],\ \theta=\frac{\pi\varepsilon_0}{\beta_0^2\Omega^4}\sigma_\eta^2 \quad (55)$$

This is the output variance for the dynamic system with nonlinear damping.

Fig. 3 shows the plots of eqn (55) (curve 2) and eqn (54) (curve 1) as functions of σ_η^2. Curve 1 is for the case $\sigma_\psi^2=1$; $\alpha=1$; $\Omega^2=25$; $\beta_0=0.2$; $\varepsilon_0=0.5$.

Example 5: Stochastic parametric excitations. Consider the special case of stochastic nonlinearity as stochastic parametric excitations. The product of two stochastic stationary processes $X_1(t)$ and $X_2(t)$ is defined as output $y(t)$ of a stochastic multiplier:

$$y(t)=kX_1(t)X_2(t) \quad (56)$$

The stochastic processes $X_1(t)$ and $X_2(t)$ are assumed to be (special case) jointly Gaussian random variables with variances σ_1^2 and σ_2^2, correlation coefficient ρ and zero means.

The covariance matrix **M** whose elements are defined by $m_{ij}=\mathbf{M}[(x_i-m_i)(x_j-m_j)]$ is:

$$M=\sigma_1\sigma_2\begin{bmatrix}\sigma_1/\sigma_2 & \rho_0 & 0 & \lambda_0\\ \rho_0 & \sigma_2/\sigma_1 & -\lambda_0 & 0\\ 0 & -\lambda_0 & \sigma_1/\sigma_2 & \rho_0\\ \lambda_0 & 0 & \rho_0 & \sigma_2/\sigma_1\end{bmatrix} \quad (57)$$

and its determinant is

$$|\mathbf{M}|=\sigma_1^2\sigma_2^2(1-\rho_0^2-\lambda_0^2)^2=\sigma_1^2\sigma_2^2(1-\rho^2)^2 \quad (58)$$

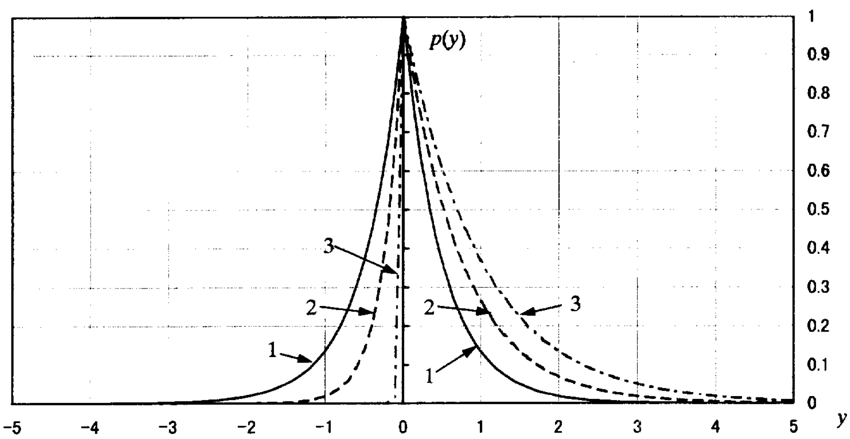

Fig. 4. Graphs of the probability density function for correlated stochastic processes.

where $\rho^2 = \rho_0^2 + \lambda_0^2 \leq 1$, which is the normalized cross-correlation coefficient.The inverse matrix of $\mathbf{M}$ is:

$$\mathbf{M}^{-1} = \frac{1}{\sigma_1\sigma_2(1-\rho^2)} \begin{bmatrix} \sigma_2/\sigma_1 & -\rho_0 & 0 & -\lambda_0 \\ -\rho_0 & \sigma_1/\sigma_2 & \lambda_0 & 0 \\ 0 & \lambda_0 & \sigma_2/\sigma_1 & -\rho_0 \\ -\lambda_0 & 0 & -\rho_0 & \sigma_1/\sigma_2 \end{bmatrix} \tag{58-a}$$

Then the probability density function of the multipliery output in eqn (56) is defined.[14,15]

$$p(y) = \begin{cases} \dfrac{1}{k\sigma_1\sigma_2} \exp\left[\dfrac{2(\rho_0 - \sqrt{1-\lambda_0^2})}{k\sigma_1\sigma_2(1-\rho^2)} y\right], & y \geq 0 \\ \dfrac{1}{k\sigma_1\sigma_2} \exp\left[\dfrac{2(\rho_0 + \sqrt{1-\lambda_0^2})}{k\sigma_1\sigma_2(1-\rho^2)} y\right], & y < 0 \end{cases} \tag{59}$$

In a special case where $k=1$, $\sigma_1^2 = \sigma_2^2 = \sigma^2$ and $\lambda_0 = 0$:

$$p(y) = \begin{cases} \dfrac{1}{\sigma^2} \exp\left[-\dfrac{2y}{\sigma^2(1+\rho)}\right], & y \geq 0 \\ \dfrac{1}{\sigma^2} \exp\left[\dfrac{2y}{\sigma^2(1-\rho)} y\right], & y < 0 \end{cases} \tag{60}$$

Fig. 4 shows the probability density function in accordance with eqn (60), where curve 1 is for $\rho = 0$, curve 2 is for $\rho = 0.5$ and curve 3 is for $\rho = 0.95$. For the correlated stochastic processes $X_1(t)$ and $X_2(t)(\rho \neq 0)$, the probability density function differs from the Gaussian (normal, $\rho = 0$) probability density function. The influence of the stochasitic parametric multiplier on the statistical moments of the linearized dynamic system output is demonstrated.

Example 6. This pertains to the correlation method for a linear system with stochastic correlated parametric excitations. Consider the dynamic system as:

$$\ddot{x} + [2\beta_0 + 2\sigma a(t)]\dot{x} + \Omega_0^2[1 + 2\mu\chi(t)]x = \nu\eta(t) \tag{61}$$

where $a(t)$, $\chi(t)$ and $\eta(t)$ are stationary δ-correlated Gaussian (statistically independent) random time functions with known statistical characteristics and zero means. Rewrite eqn (61) as:

$$\ddot{x} + 2\beta_0\dot{x} + \Omega^2 x = \nu\eta(t) - 2\sigma a(t)\dot{x} - 2\mu\Omega^2\chi(t)x \tag{62}$$

Fig. 5(a) shows the block diagram of the dynamic system described in eqn (62). The solution of eqn (62) is assumed as $x(t) = x_0(t) + x_1(t)$, where $x_0(t)$ is the solution of the system with constant parameters subjected to the process $\eta(t)$, and $x_1(t)$ is the solution of the system under parametric excitations $a(t)$ and $\chi(t)$. Writing the equation of motion at two time moments t and $t = \tau$ and using the Duhamel's integral after averaging M[x(t)x(τ)] for variance $\sigma_x^2 = R_x(\tau=0) = M[x(t)x(t+\tau)]$ in the stationary regime, one obtains:

$$\sigma_x^2 = \frac{\sigma_\eta^2 \int_0^\infty k^2(\theta)\,\mathrm{d}\theta}{1 - [\sigma_\chi^2 \int_0^\infty k^2(\theta)\,\mathrm{d}\theta + \sigma_a^2 \int_0^\infty \delta^{(1)}(\xi)\,\mathrm{d}\xi \int_0^\infty \delta^{(1)}(n)\,\mathrm{d}n \int_0^\infty k(\theta+n-l)k(l)\,\mathrm{d}l]} \tag{63}$$

In a special case where $\sigma_a^2 = 0$:

$$\sigma_x^2 = \frac{\sigma_\eta^2 \int_0^\infty k^2(\theta)\,\mathrm{d}\theta}{1 - \sigma_\chi^2 \int_0^\infty k^2(\theta)\,\mathrm{d}\theta} \tag{64}$$

where

$$\sigma_\eta^2 = \nu^2;\ \sigma_\chi^2 = 4\mu^2\Omega^4;\ \delta^{(1)}(t) = \frac{\mathrm{d}\delta(t)}{\mathrm{d}t};$$

$$k(\tau) = \frac{1}{\Omega_1} e^{-\beta_0\tau} \sin\Omega_1\tau;\ \Omega_1 = \sqrt{\Omega^2 - \beta_0^2};\ \sigma_a^2 = 4\sigma^2 \tag{64a}$$

and $\delta(t)$ is the generalized Dirac function.

Using the probability density function $P_{\mathrm{st.l}}(A)$ for the stationary regime given in Part 1 of this paper:[1]

$$p_{\mathrm{st.l}}(A) = \frac{A}{\sigma_1^2} \exp\left(-\frac{A^2}{2\sigma_1^2}\right) \tag{64b}$$

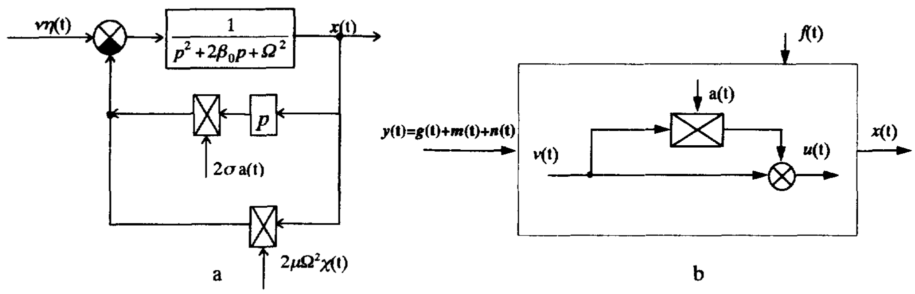

Fig. 5. Structure of linear system with random parametric excitations.

the second statistical moment $\langle A^2\rangle$ can be defined as

$$\langle A^2\rangle = \int_0^\infty A^2 p_{\mathrm{st.1}}(A)\,\mathrm{d}A = \int_0^\infty \frac{A^3}{\sigma_1^2}\exp\left\{-\frac{A^2}{2\sigma_1^2}\right\}\mathrm{d}A$$

$$= 2\sigma_1^2 = \frac{\nu^2}{2\Omega^2\left(\beta_0 - \frac{3}{4}\mu^2\Omega^2\right)} \tag{65}$$

From eqn (64), one obtains:

$$\sigma_x^2 = \frac{\nu^2\int_0^\infty k^2(\tau)\,\mathrm{d}\tau}{(1-4\mu^2\Omega^4\int_0^\infty k^2(\tau)\,\mathrm{d}\tau)} = \frac{\nu^2}{2\Omega^2(\beta_0-\mu^2\Omega^2)} \tag{66}$$

From eqns (63) and (64), the parametric random excitations by linear terms x and $\dot{x}$ lead to the increase in variance of the motion of the dynamic systems. Fig. 3 and analysis of eqn (54) indicated that the random excitations in stochastic nonlinearity lead to the decrease in the variance of motion.

Remark 2. In Ref. [16], the problem of decrease in the resonance amplitudes due to random parametric excitations in linear term $x(t)$ is discussed. The parametric excitations are assumed of short durations. Eqn (54) shows a new possibility of the decrease in the variance of output.[2,17]

In general, the output of the linearized system with random parametric excitations is described as:

$$x(t) = x_0(t) + \int_0^t a(\theta)\nu(\theta)k_{xu}(t-\theta)\,\mathrm{d}\theta \tag{67}$$

where $k_{xu}(t)$ is an impulse transfer function from point u to point x in the block diagram of the dynamic system [see Fig. 5(b)] and a random process $\nu(t) = \{x, \dot{x},...\}$ has the cross-correlation function $R_{a\nu}(t, \theta)$ with random process $a(t)$, and a random process $a(t)$ has the autocorrelation function $R_a(t, \theta)$. From eqn (67), the mean $m_x(t)$ and autocorrelation functions $R_x(t_1, t_2)$ of output $x(t)$ are obtained as:

$$m_x(t) = \left\langle x_0(t) + \int_0^t a(\theta)\nu(\theta)k_{xu}(t-\theta)\,\mathrm{d}\theta\right\rangle$$

$$= m_{x_0}(t) + \int_0^t R_{a\nu}(\theta,\theta)k_{xu}(t-\theta)\,\mathrm{d}\theta \tag{68}$$

and

$$\begin{aligned}
R_x(t_1,t_2) = \Big\langle &(x_0(t_1) + \int_0^{t_1} a(\theta_1)\nu(\theta_1)k_{xu}(t_1-\theta_1)\,\mathrm{d}\theta_1)\\
&\times (x_0(t_2) + \int_0^{t_2} a(\theta_2)\nu(\theta_2)k_{xu}(t_2-\theta_2)\,\mathrm{d}\theta_2\Big\rangle\\
= R_{x_0}&(t_1,t_2) + \int_0^{t_1}[R_{x_0u}(\theta,t_2) + m_\nu(\theta)R_{x_0a}(\theta,t_2)]\\
&\times k_{xu}(t_1-\theta)\,\mathrm{d}\theta + \int_0^{t_2}[R_{ux_0}(t_1,\theta) + m_\nu(\theta)\\
&\times R_{ax_0}(t_1,\theta)]k_{xu}(t_2-\theta)\,\mathrm{d}\theta\\
&+ \int_0^{t_1} k_{xu}(t_1-\theta_1)\,\mathrm{d}\theta_1\\
&\times \int_0^{t_2}[m_\nu(\theta_1)R_{ua}(\theta_1,\theta_2) + m_\nu(\theta_2)R_{au}(\theta_1,\theta_2)]\\
&\times k_{xu}(t_2-\theta_2)\,\mathrm{d}\theta_2 + \int_0^{t_1} k_{xu}(t_1-\theta_1)\,\mathrm{d}\theta_1\\
&\times \int_0^{t_2}[m_\nu(\theta_1)m_\nu(\theta_2)R_a(\theta_1,\theta_2)\\
&+ R_\nu(\theta_1,\theta_2)R_a(\theta_1,\theta_2) + R_{a\nu}(\theta_1,\theta_2)R_{\nu a}(\theta_1,\theta_2)]\\
&\times k_{xu}(t_2-\theta_2)\,\mathrm{d}\theta_2
\end{aligned} \tag{69}$$

where

$$R_{a\nu}(t_1,t_2) = R_{a\nu_0}(t_1,t_2) + \int_0^{t_1} R_{au}(\theta,t_2)k_{\nu u}(t_1-\theta)\,\mathrm{d}\theta$$

$$+ \int_0^{t_1} m_\nu(\theta)R_a(\theta,t_2)k_{\nu u}(t_1-\theta)\,\mathrm{d}\theta \tag{70}$$

is the cross-correlation function of processes $a(t)$ and $\nu(t)$ and m_ν, and $R_\nu(t_1, t_2)$ are the mean and autocorrelation function of process $\nu(t)$, respectively. The functions m_ν and $R_\nu(t_1, t_2)$ are obtained from eqns (68)–(70) by changing x to ν.

Remark 3. eqns (68) and (69) is defined with the calculation of the third centered statistical moments as:

$$R_{qu}(t_1,t_2) = M[a(t_1)\nu^0(t_1)q(t_2)] \tag{71}$$

The value of cross-correlated function $R_{qu}(t_1, t_2)$ is calculated under the assumption that processes $q(t)$ and $a(t)$ are random functions with zero means as:

$$R_{qu}(t_1,t_2)=\int_0^{t_1}[R_a(t_1,\theta)R_{qv}(\theta,t_2)+R_{aq}(t_2,\theta)R_{va}(t_1,\theta)]\times k_{vu}(t_1-\theta)\,\mathrm{d}\theta \quad (72)$$

For $q(t) = a(t)$:

$$R_{au}(t_1,t_2)=\int_0^{t_1}[R_a(t_1,\theta)R_{av}(\theta,t_2)+R_a(t_2,\theta)\times R_{va}(t_1,\theta)]k_{vu}(t_1-\theta)\,\mathrm{d}\theta \quad (73)$$

The cross-correlation function $R_{qv}(t_1, t_2)$ in eqn (72) is defined as:

$$R_{qv}(t_1,t_2)=R_{qv_0}(t_1,t_2)+\int_0^{t_1}[m_v(\theta)R_{qu}(\theta,t_2)+R_{qu}(\theta,t_2)]\times \dot{k}_{vu}(t_1-\theta)\,\mathrm{d}\theta \quad (74)$$

Therefore, eqn (72) becomes:[17,18]

$$\begin{aligned}R_{qu}(t_1,t_2)=&\int_0^{t_1}[R_{qa}(\theta,t_1)R_{va}(t_1,\theta)+R_a(t_1,\theta)R_{qv_0}(\theta,t_2)]\\&\times k_{vu}(t_1-\theta)\,\mathrm{d}\theta+\int_0^{t_1}R_a(t_1,\theta)k_{vu}(t_1-\theta)\,\mathrm{d}\theta\\&\times\int_0^{\theta}[m_v(\tau)R_{qa}(\tau,t_2)\\&+R_{qu}(\tau,t_2)]k_{vu}(\theta-\tau)\,\mathrm{d}\tau\end{aligned} \quad (75)$$

The central moments of the fourth order are expressed in eqns (69), (73) and (75) through the statistical moments of the second order under the assumption that random functions are with the normal probability density functions. Eqns (70)–(72) and (75) form a closed system of integral equations.

From eqns (68) and (69), it is demonstrated that the behavior of means and autocorrelation functions in the linearized system with stochastic parametric excitations depends on external input excitations, random parametric processes and the cross-correlation between input processes.

Special Case. If process $a(t) = a$ is a random variable with variance σ_a^2 and input processes are uncorrelated with random value a, then:

$$m_x(t)=m_{x_0}(t)+m_{v_0}\frac{\sigma_a^2\int_0^{\infty}k_{xu}(\theta)\,\mathrm{d}\theta\int_0^{\infty}k_{vu}(\tau)\,\mathrm{d}\tau}{1-3\sigma_a^2[\int_0^{\infty}k_{vu}(\theta)\,\mathrm{d}\theta]^2} \quad (76)$$

and

$$\begin{aligned}S_x(\omega)=&S_{x_0}(\omega)+2\pi C_1(\int_0^{\infty}k_{xu}(\theta)\,\mathrm{d}\theta)^2\delta(\omega)\\&+\sigma_a^2\Bigg\{S_v(\omega)|\Phi_{xu}(j\omega)|^2\\&+\frac{S_{x_0v_0}(\omega)\Phi_{vu}(j\omega)\Phi_{xu}(j\omega)}{1-\sigma_a^2\Phi_{vu}^2(j\omega)}\\&+\frac{S_{\gamma_0x_0}(\omega)\Phi_{vu}^*(j\omega)\Phi_{xu}^*(j\omega)}{1-\sigma_a^2\Phi_{vu}^{*2}(j\omega)}\Bigg\}\end{aligned} \quad (77)$$

where

$$C_1=m_v^2\sigma_a^2+R_{av}(0)+2m_vR_{au}(0);$$

$$R_{au}(0)=2\sigma_a^2\int_0^{\infty}k_{vu}(\theta)\,\mathrm{d}\theta \quad (77a)$$

and[18]

$$R_{av}(0)=m_{v_0}\left[\frac{\sigma_a^2\int_0^{\infty}k_{vu}(\theta)\,\mathrm{d}\theta}{1-3\sigma_a^2(\int_0^{\infty}k_{vu}(\theta)\,\mathrm{d}\theta)^2}\right] \quad (78)$$

In eqns (76) and (77), the first terms represent the statistical moments in the system with constant parameters and the second terms describe the change of statistical moments as functions of parametric excitations. Calculation of the third-order statistical moment indicate the dependence of the statistical output characteristics from the correlation between the frequency response of system and input excitations.

Example 7. Consider a simple dynamic system described as:

$$\Phi(s)=\frac{10}{s(0.1s+1)};\ S_y(\omega)=0.4\tau^3\frac{\omega^2}{\tau^4\omega^4-1.92\tau^2\omega^2+1} \quad (79)$$

where $\Phi(s)$ is the transfer function of dynamic system and $S_y(\omega)$ is a power spectral density of stochastic input process.

Fig. 6 plots the output increment variance $\Delta\sigma_x^2$ (pure random parametric excitation) calculated using eqns (77) and (78) with $\sigma_a^2=0.1$ (curve 1) and $\sigma_a^2=0.2$ (curve 2), with the statistical moment of the third order. Curves 3 and 4 represented the results for $\sigma_a^2=0.1$ and 0.2 without the third statistical moment. Without calculating the statistical moment of the third order as shown in the figure, the increment of output variance is always positive. The calculation of the statistical moment of third order[2,18] gives both a positive and a negative value of increment $\Delta\sigma_x^2$. This is consistent with the result obtained on the basis of the FPK equation in Part 1 of this paper.[1]

4.2 Case 2: nonlinear dynamic system with jump stochastic parametric excitations

Consider a dynamic system described by a stochastic differential equation:

$$\frac{\mathrm{d}x_k}{\mathrm{d}t}=\sum_{l=1}^{n}[a_{kl}(t)+z_{kl}(t)+\varphi_{kl}(x,t)]F_{kl}(x)+N_k(t) \quad (80)$$

where a stochastic process $\varphi_{kl}(x, t)$ describes a stochastic jump change of the characteristics of the system; processes $a_{kl}(t)$, $z_{kl}(t)$ and $N_{kl}(t)$ are the same as in Section 4.1.

A random change of characteristics in the dynamic system described as eqn (80) is a stochastic event that is independent of the system motion. In this case, eqn (80)

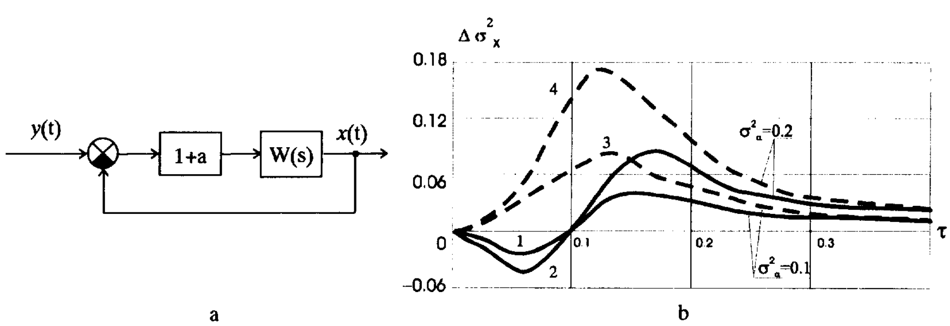

Fig. 6. Graphs of relationship $\Delta\sigma_x^2 = f(\sigma_\alpha^2)$: (a) block diagram of dynamic system; (b) output increment variance $\Delta\sigma_x^2(\sigma_\alpha^2, \tau)$.

describes the behavior of the dynamic system with possible characteristics violations[19] or with random change of parameters.[2] Another case is that the stochastic change of characteristics depends on the phase coordinate of system output. Both cases defined as:

$$\varphi(x,t) = \begin{cases} \varphi_1(t) \\ \varphi_{2i}(x), \; |x| = c_i \end{cases} \tag{81}$$

Consider the first case of eqn (81) when $\{x(t);\varphi(x,\,t) = \varphi_1(t) = \varphi_{kl}(t)\}$. The random process $\varphi_{kl}(x,t)$ has a final number of value state $\varphi_{kl}(t) = \{\varphi_{kl}^{(1)}(t),\, \varphi_{kl}^{(2)}(t), \ldots, \varphi_{kl}^{(q)}(t)\}$ and is subjected to the Poisson probability distribution law. The probability $p^{(ba)}(t)$ of transfer from state (a) to state (b) in a time duration $(t,\, t + \Delta t)$ is:

$$p^{(ba)}(t) = \nu^{(ba)}(t)\Delta t + 0(\Delta t);\; b \neq a(a, b = 1, 2, \ldots, q);$$
$$p^{(aa)}(t) = 1 - \nu^{(aa)}(t)\Delta t + 0(\Delta t);\; \nu^{(aa)} = \sum_{b\neq a} \nu^{(ba)};$$
$$\lim_{\Delta\to 0} \frac{0(\Delta t)}{\Delta t} = 0 \tag{82}$$

eqn (80) is a stochastic differential equation with jump stochastic parametric excitations described as eqn (82). The joint pair $\{x(t);\varphi_{kl}(t)\}$ form the Markovian stochastic process and all the probability characteristics can be evaluated using the FPK equation.[2]

The probability density function $w(x,\, t)$ for vector $\mathbf{x}$ is defined as

$$w(x,t) = \sum_{a=1}^{q} w^{(a)}(t) \tag{83}$$

The FPK equation for evaluating $w^{(a)}(x,\, t)$ is written (see Appendix A) as:[2,20]

$$\frac{\partial w^{(a)}(x,t)}{\partial t} = (L^{(a)} - \nu^{(aa)})w^{(a)}(x,t) + \sum_{b\neq a} \nu^{(ab)} w^{(b)}(x,t) \tag{84}$$

where

$$L^{(a)} = -\sum_{k=1}^{n} \frac{\partial}{\partial xk} A_k^{(a)} + \frac{1}{2}\sum_{k,l=1}^{n} \frac{\partial^2}{\partial x_k \partial x_l} B_{kl}^{(a)} \tag{85}$$

and function $w^{(a)}(x,\, t)$ satisfies the normalization condition

$$\sum_{a=1}^{q} \int w^{(a)}(x,t)\, \mathrm{d}x = 1 \tag{86}$$

In general, the system in eqn (84) includes m partial differential equations of the second order and it is possible to obtain exact solutons only for simple cases.[2,20] For engineering analysis, eqn (84) can be used to calculate mixed moments or cumulants of the system phase coordinates.

Example 8. This pertains to the method of moments for the statistical analysis of nonlinear dynamic systems with random jump parametric excitations. Multiply eqn (84) with $x_1^{r_1} x_2^{r_2}, \ldots, x_n^{r_n}$and integrate it over the full phase space to obtain the following system of equations for mixed moment $\alpha_{r_1 r_2,\ldots,r_n}^{(a)}(t)$ of order N as:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\alpha_{r_1 r_2,\ldots,r_n}^{(a)}) = &\left\{ \sum_{k=1}^{n} r_k \left\langle (A_k^{(a)} x_k^{r_k-1} \prod_{p=1}^{n} x_p^{r_p})|a \right\rangle \right. \\ &+ \sum_{k=1}^{n} r_k(r_k-1)\left\langle (B_{kk}^{(a)} x_k^{r_k-2} \prod_{p=1}^{n} x_p^{r_p}|a \right\rangle \\ &+ \sum_{k,\nu=1, k\neq \nu}^{n} r_k r_\nu \\ &\times \left. \left\langle (B_{k\nu}^{(a)} x_k^{r_k-1} x_\nu^{r_\nu-1} \prod_{p=1}^{n} x_p^{r_p}|a \right\rangle \right\} p^{(a)}(t) \\ &+ \sum_{b\neq a} [\nu^{(ab)} \alpha_{r_1 r_2,\ldots,r_n}^{(b)} - \nu^{(ba)} \alpha_{r_1 r_2,\ldots,r_n}^{(a)}]\end{aligned} \tag{87}$$

where $\sum_{i=1}^{N} r_i = N > 0$.The probability $p^{(a)}(t)$ that parameters of system are in state (a) is:

$$p^{(a)}(t) = \int_{-\infty}^{\infty} w^{(a)}(x,t)\, \mathrm{d}x \tag{88}$$

and

$$\frac{\mathrm{d}p^{(a)}(t)}{\mathrm{d}t} = -\nu^{(aa)} p^{(a)}(t) + \sum_{b\neq a} b^{(ba)} p^{(b)}(t) \;(a = 1,\, 2, \ldots, r) \tag{89}$$

with the initial conditions as $p^{(a)}(t_0)=p_0^{(a)}$. The solution of eqn (89) is subject to the condition of normalization as $\sum_a p^{(a)}(t)=1$.

If $\nu^{(aa)}(t)$ = *constant* (as, for instance, in safety theory), then eqn (89) can be solved. In eqn (87), take into consideration that:

$$\alpha^{(a)}_{r_1r_2,\ldots,r_n}=\int_{(n)}x^r\omega^{(a)}(x,t)\,\mathrm{d}x=\alpha_{r_1r_2,\ldots,r_n}(t|a)p^{(a)}(t) \tag{90}$$

and the absolute mixed moment is defined as

$$\alpha_{r_1r_2,\ldots,r_n}=\sum_a\alpha^{(a)}_{r_1r_2,\ldots,r_n}(t) \tag{91}$$

If $r_i=0$ for $i\neq k$ and $r_k=1$, then eqn (87) becomes:

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha_{0,\ldots,r_k=1,\ldots,0}=\frac{\mathrm{d}}{\mathrm{d}t}m_k^{(a)}=\langle A_k^{(a)}(t,x)|a\rangle p^{(a)}(t)+\sum_{b\neq a}[\nu^{(ab)}m_k^{(b)}(t)-\nu^{(ba)}m_k^{(a)}(t)] \tag{92}$$

where $m_k=\langle x_k\rangle$ with initial conditions as

$$m_k^{(a)}(t_0)=m_{k0}^{(a)}=\int x_kw_0^{(a)}(x)\,\mathrm{d}x \tag{93}$$

For the second moments, $r_i=0$ for $i\neq k$, $i\neq\nu$ and $r_k=r_\nu=1$, it becomes:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\alpha_{0,\ldots,r_k=1,\ldots,r_\nu=1,\ldots,0}&=\frac{\mathrm{d}}{\mathrm{d}t}\theta^{(a)}_{k\nu}\\&=\langle(x_\nu A_k^{(a)}+x_kA_\nu^{(a)}+2B_{k\nu}^{(a)}|a\rangle p^{(a)}(t)\\&\quad+\sum_{b\neq a}[\nu^{(ab)}\theta_{k\nu}^{(b)}(t)-\nu^{(ba)}\theta_{k\nu}^{(a)}(t)]\end{aligned} \tag{94}$$

The initial conditions for eqn (94) are similar to eqn (93). In the special case $\nu^{(ij)}=0$ from eqn (88), $p^{(a)}(t)=1$. From eqns (92) and (94), eqn (49) is obtained. From eqn (87), eqn (50) is obtained.

Special Case. This case is a linear system with jump stochastic parameters. A general class of linear systems in which the variation of parameters affect both free and forced motion can be described by the following vector equation:

$$\dot{\mathbf{y}}=\mathbf{E}(t)\mathbf{y}(t)+\mathbf{D}(t)\boldsymbol{\xi}(t)+\mathbf{C}(t)\mathbf{x}(t) \tag{95}$$

where $\mathbf{y}=\{y_1(t),\ y_2(t),\ldots,y_n(t)\}^{\mathrm{T}}$ is a column vector describing the coordinates of an nth-order system; $\mathbf{E}(t)=\|E_{ik}(t)\|$ is an $n\times n$ matrix characterizing the free motion of the system and is determined by its characteristics; $\xi(t)$ and $\mathbf{x}(t)$ are vectors defining the random and nonrandom input effects and having no more than n dimensions; $\mathbf{D}(t)=\|D_{ik}(t)\|$ and $\mathbf{C}(t)=\|C_{ik}(t)\|$ are matrices of a suitable order characterizing the forced motion and determined by the system characteristics. The vector $\xi(t)$ will be considered to be a δ-correlated stochastic process, i.e.:

$$\langle\boldsymbol{\xi}(t_1)\boldsymbol{\xi}^{\mathrm{T}}(t_2)\rangle=\|S_{ik}\|\delta(t_1-t_2) \tag{96}$$

where $\mathbf{S}=\|S_{ik}\|$ is the matrix of the spectral densities and T is the sign of the transformation.

The random functions of time $\mathbf{E}(t)$, $\mathbf{D}(t)$ and $\mathbf{C}(t)$ may assume only a finite number of values $E^{(1)}(t),\ldots,E^{(m)}(t)$; $D^{(1)}(t),\ldots,D^{(m)}(t)$; and $C^{(1)}(t),\ldots,C^{(m)}(t)$ and are subject to the Poisson rules. So the process $y(t)$ is a rather complex composition of diffusion process with Poisson transitions between separate elements of the composition. The set of $y(t)$, $\mathbf{E}(t)$, $\mathbf{D}(t)$ and $\mathbf{C}(t)$ forms a Markovian process, and consequently all probability properties of $y(t)$ can be precisely determined. In this case, drift $A^{(a)}$ and diffusion $B^{(a)}$ coefficients can be determined for eqn (95) as:

$$A^{(a)}=E^{(a)}y+C^{(a)}x;\ B^{(a)}=D^{(a)}SD^{(a)T} \tag{97}$$

respectively.The first moment function is determined as $m^{(a)}(t)=\int yw^{(a)}(y,t)\,\mathrm{d}y=m(t|a)p^{(a)}$ and the probability unconditional expected value of the coordinate y is determined by:

$$m(t)=\langle y\rangle=\sum_a m^{(a)}(t) \tag{98}$$

The ordinary differential equation for determing the value of $m(t)$ is:

$$\dot{m}^{(a)}(t)=(E^{(a)}-\nu^{(aa)})m^{(a)}+\sum_{b\neq a}m^{(b)}(t)+C^{(a)}xp^{(a)}\ (a=1,\ldots,m) \tag{99}$$

with initial conditions

$$m^{(a)}(t_0)=m_0^{(a)}=\int yw_0^{(a)}(y)\,\mathrm{d}y \tag{99a}$$

The unconditional matrix of the second moment for the system coordinate y is defined by:

$$K(t)=\|K_{ik}(t)\|=\langle yy'\rangle=\sum_a\mathbf{K}^{(a)}(t), \tag{100}$$

$$\mathbf{K}^{(a)}(t)=\|K_{ik}^{(a)}(t)\|=\int yy^{\mathrm{T}}w^{(a)}(y,t)\,\mathrm{d}y=K(t|a)p^{(a)}(t)$$

where $\mathbf{K}^{(a)}(t)$ is a system of second-moment functions and the correlation matrix $\mathbf{R}(t)$ is defined as $\mathbf{R}(t)=K(t)-m(t)m^{\mathrm{T}}(t)$.Matrix $\mathbf{K}^{(a)}$ is calculated from the following equation[20]

$$\dot{K}^{(a)}=E^{(a)}\mathbf{K}^{(a)}+\mathbf{K}^{(a)}E^{(a)T}+B^{(a)}p^{(a)}-\nu^{(aa)}\mathbf{K}^{(a)}+\sum_{b\neq a}\nu^{(ab)}\mathbf{K}^{(b)}+m^{(a)}x^{\mathrm{T}}C^{(a)T}+C^{(a)}xm^{(a)T} \tag{101}$$

The initial conditions for eqn (101) are found from the initial conditions for the FPK equations for special case $y(t_0)=y_0$, and parameters of system are in state (c):

$$\mathbf{K}^{(c)}(t_0)=y_0y_0^{\mathrm{T}},\ \mathbf{K}^{(a)}(t_0)=0\ (a\neq c,\ a=1,\ldots,m) \tag{101a}$$

eqn (101) is a set of $m[n(n+1)]/2$ ordinary differential equations that can be solved by the standard methods. In Part 3 of this paper, the results of stochastic simulation and a study on the stochastic stability conditions for this class of systems will be described.

5 STATISTICAL ANALYSIS OF TIME-VARIANT DYNAMIC SYSTEMS ON BASIS OF FPK EQUATION

A nonlinear dynamic system subject to external random excitations is described by a set of stochastic differential equations:

$$\dot{x}_i = F_i(x,t) + \sum_{j=1}^{r} a_{ij}(x)\xi_j(t) \; (i = 1, 2, \ldots, n) \tag{102}$$

$F_i(x)$ are piecewise-differentiable and a_{ij} are piecewise-constant.

The phase space $x = (x_1, x_2, \ldots, x_n)^T$ is partitioned by a finite number of hyperplanes of the form:

$$\langle cx \rangle = 0 \tag{103}$$

where $c = (c_1, c_2, \ldots, c_n)^T$ is a vector normal to the hyperplane (104), and the angle brackets denote the scalar product of the vector.

Continuity is upset only at a finite number of hyperplane (103), where the functions in eqn (102) can have first-order discontinuities. In the event $\xi_j(t)$ are independent Gaussian stationary white noise processes with $M[\xi_j(t)] = 0$ and $M[\xi_j(t)\xi_j(t+\tau)] = \varepsilon_j(t)\delta(\tau)$ and eqn (102) can be analyzed by means of the FPK equations for diffusion-type Markovian processes. In this case, in each of the continuity of the functions $F_i(x)$ and $a_j(x)$, eqn (102) defines a multidimensional Markovian process $x(t) = \{x_1(t), x_2(t), \ldots, x_n(t)\}^T$, which has a corresponding probability density $w(x, t)$ satisfying the familiar FPK equation.

$$\frac{\partial w}{\partial t} = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\{A_i w\} + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}\{B_{ij} w\} \tag{104}$$

where

$$A_i(x,t) = F_i(x,t) + \frac{1}{2}\sum_{s=1}^{n}\sum_{j=1}^{r} \frac{\partial a_{ij}}{\partial x_s} a_{sj}\varepsilon_j \tag{105a}$$

$$B_{ij}(x,t) = \sum_{k=1}^{r} a_{ik}a_{jk}\varepsilon_j \tag{105b}$$

are, respectively, a component of the drift vector and an element of the diffusion matrix of the multidimensional Markovian process as in eqn (45).

The determination of the density function $w(x, t)$ throughout the entire phase space of the system involves the generation of boundary conditions to match the function $w(x, t)$ satisfying eqn (104) in each of the domains set apart by the hyperplanes [eqn (103)].

Remark 4. The boundary conditions have been derived in Refs [21,22] and given in a more general form in Ref.[17] (see Appendix B). The coefficients $a_{ij}(x)$ in eqn (102) may be constants throughout the entire phase plane or have first-order discontinuities, depending on the specific flow diagram of the dynamic system and the location of noise entry. In the latter case, knowledge of the local characteristics of the Markovian process $x(t)$ on both sides of the surface of discontinuity of the coefficients is inadequate for determining the boundary conditions for eqn (104), and instead it is required to analyze the behavior of the process on the surface itself. Under the reasonable assumption that $w(x, t)$ does not have δ-singularities, the matching conditions[22] have the general form:[17]

$$\sum_{i,j=1}^{n} c_i c_j \Delta[B_{ij} w(x,t)] = 0 \tag{106}$$

$$div_r\left[-\frac{1}{2}\sum_{j=1}^{n} c_j \Delta[B_{ij} w(x,t)\right] + \sum_{i=1}^{n} c_i \Delta\left[A_i w(x,t) - \frac{1}{2}\frac{\partial}{\partial x_i} B_{ij} w(x,t)\right] = 0 \tag{107}$$

where $\delta[f] = f(\nu = 0_+) - f(\nu = 0_-)$ and $div_r[.]$ is $(n-1)$-dimensional divergence on the hyperplane [eqn (103)].

Eqns (106) and (107) are described in details in Appendix B. The first condition [eqn (106)] represents the continuty of probability density $w(x, t)$ and the second condition [eqn (107)] represents the continuty of flux density across a surface of discontinuity described by eqn (103).

The FPK equation is also considered to be valid when the pertubing influence is not white noise, but a stationary Gaussian process with a rapidly damped correlation function $R(\tau)$, and with a sufficiently wide bandpass (the correlation time τ_0 is small in comparasion with the setting time of the transient processes). In this case, the stationary Gaussian process may be replaced by equivalent white noise with a correlation function $R_f^0(\tau) = C_f^2\delta(\tau)$, where $C_f^2 = \int_0^\infty R_f(\tau)\,d\tau$ and τ_0 may be approximately estimated from the following relationship for a Gaussian process:[23]

$$\tau_0 = 1/(R_f(0))\int_0^\infty R_f(\tau)\,d\tau = (C_f^2)/(R_f(0))$$

With the foregoing remarks in mind, it is possible to perform statistical analysis of time-variant dynamic systems described by eqns (102) and (103) for the following two cases.

5.1 Case 1: dynamic system with jump irreversible time-variant characteristics

Consider the definition of a probability density function $P(x, t)$ for a piecewise-linear system described by a set of equations:

$$x = x_1, \quad \frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = -2\alpha x_2 - \omega^2(x_1, t)x_1 + f(t) \tag{108}$$

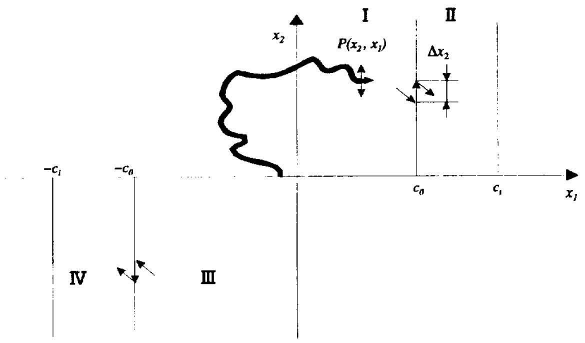

Fig. 7. Phase plane of dynamic system with jump time-variant characteristics.

where

$$\omega^2(x_1,t)=\begin{cases}\omega_0^2, \text{ until } |x_1|<c_0,\\ \omega_i^2, \text{ for } |x_1|\geq c_i \vee |x_1|<c_{i+1}\\ (i=0,\ 1,\ldots,n)\end{cases} \tag{108a}$$

and $f(t)$ is white noise ($R_f(\tau)=C_f^2\delta(\tau)$).The surface of discontinuity eqn (103) for eqn (108) is described as:

$$|x_1|=c_i \tag{109}$$

The lines in eqn (109) are the switching line of the system characteristics of eqn (108).

Remark 5. eqn (108) describes a class of time-variant dynamic systems and provides the theoretical background for designing seismic resistant structures with switching couplings.[2,3] In this case, by switching internal couplings, jumping change of the natural frequencies of the structure is achieved. The frequency accomplishes a random jump $h_i=\omega_{i+1}^2-\omega_i^2$ $(\omega_{i+1}^2<\omega_i^2)$ of negative sign in a random time moment when parameters of system motion achieve or exceed any constant given in eqn (109).

For eqns (108) and (109), the FPK eqn (104) can be written ($w\equiv p$) as

$$\begin{aligned}L[P(x_1,x_2,t)]=&-\frac{\partial P}{\partial t}-\frac{\partial}{\partial x_1}\{x_2P\}\\&-\frac{\partial}{\partial x_2}\{[-2\alpha x_2-\omega_i^2x_1]P\}\\&+\frac{C_f^2}{2}\frac{\partial^2 P}{\partial x_2^2}=0, \text{ until } |x_1|<c_i\end{aligned} \tag{110}$$

and

$$\begin{aligned}L[P(x_1,x_2,t)]=&-\frac{\partial P}{\partial t}-\frac{\partial}{\partial x_1}\{x_2P\}\\&-\frac{\partial}{\partial x_2}\{[-2\alpha x_2-\omega_{i+1}^2x_1]P\}\\&+\frac{C_f^2}{2}\frac{\partial^2 P}{\partial x_2^2}=0\end{aligned}$$

$$\text{for } |x_1|\geq c_i \vee |x_1|<c_{i+1} \tag{111}$$

The phase space (x_1, x_2) of eqn (108) with the logical conditions of eqn (109) is divided into two domains as $\{1\}=\{I\}+\{II\}=\{x:\ 0\leq x_1<c_i;\ 0\leq x_2<\infty\}$ and $\{2\}=\{III\}+\{IV\}=\{x:\ -c_i<x_1\leq 0;\ -\infty<x_2\leq 0\}$. Fig. 7 shows these domains.

As an example, it is possible to consider a change of system characteristics in domain I (see Fig. 7). The diffusion coefficients B_{ij} of the two-dimensional process are discontinuous on switching line $x_1=c_0$ and the first condition of conjugation eqn (106) can be written as:

$$P^+=P^-=P_1(c_0,x_2,t) \tag{112}$$

where P^+ is a limit value of probability density function $P(x,\ t)$ when the value of x_1 approaches the switching line [eqn (109)] from the domain $x_1<c_0$, and P^- is a corresponding limit value when $P(x,\ t)$ approaches the switching line from the domain $x_1\geq c_0$.

The probability flux density for eqn (111) is:

$$x_2^-P^-+[-2\alpha x_2^- -\omega_{i+1}^2c_i]P^-+\frac{C_f^2}{2}\left(\frac{\partial P}{\partial x_2}\right)^- \tag{113}$$

The corresponding flux for eqn (110) is

$$x_2^+P^++[-2\alpha x_2^+ -\omega_i^2c_i]P^-+\frac{C_f^2}{2}\left(\frac{\partial P}{\partial x_2}\right)^+ \tag{114}$$

Subtracting eqn (113) from eqn (114) results in zero.

Therefore, the second condition of conjugation can be written as:

$$\frac{2\Delta x_2P_1(1-2\alpha)}{C_f^2}+\frac{2P_1c_ih_i}{C_f^2}+\left(\frac{\partial P}{\partial x_2}\right)^-=\left(\frac{\partial P}{\partial x_2}\right)^+ \tag{115a}$$

or

$$\left(\frac{\partial P}{\partial x_2}\right)^-=P_2-\left[\frac{2\Delta x_2(1-2\alpha)}{C_f^2}+\frac{2c_ih_i}{C_f^2}\right]P_1 \tag{115b}$$

where $P_2=(\partial P^+)/(\partial x_2)$ and $\Delta x_2=(x_2^- - x_2^+)$ is the production velocity of kinetic energy for switching internal couplings of the system.

Remark 6. A generalized form of eqns (108) and (109) is:

$$\ddot{x}+n(x,t)\dot{x}+\omega^2(x,t)x=f(t) \tag{116}$$

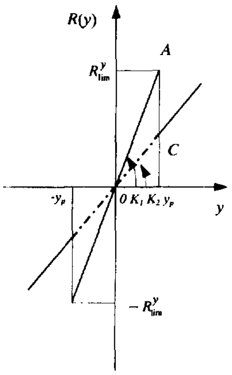

Fig. 8. 'Restoring force–displacement' for dynamic system with jump time-variant characteristics.

where $n(x, t)$ and $\omega^2(x,t)$ are irreversible parameters and

$$\omega^2(x_1,t)=\begin{cases}\omega_{is}^2, \text{ until } |x_1|<C_0,\\ \omega_{ls}^2, \text{ for } |x_1|\geq C_i \vee |x_1|<C_{i+1}\\ (i=0,\ 1,\ldots,n)\end{cases} \tag{117a}$$

$$n(x_1,t)=\begin{cases}n_{is}, \text{ until } |x_1|<C_0,\\ n_{ls}, \text{ for } |x_1|\geq C_i \vee |x_1|<C_{i+1}\\ (i=0,\ 1,\ldots,n),\end{cases} \tag{117b}$$

where, *is* and *ls* are the indices of initial and limit systems.

Figs 7 and 8 show the physics and calculation models of eqns (116) and (117) that are described in details in Ref. [2]. The time-variant characteristics (an accumulation of local destruction) are associated with specific parametric excitations as in eqn (117).

Consider the perculiarity of the 'restoring force–displacement' plot in Fig. 8. According to the energy conservation law, for part OA, $E_1=T_1+\Pi_1=(mv_1^2+C_1y_p^2)/2$; for part OC, $E_2=T_2+\Pi_2=(mv_2^2+C_2y_p^2)/2$ and $E_1 = E_2$. Let $v_2 = v_1 + \Delta v$, then from the energy conservation law:

$$\Delta x_2 = -x_2+\sqrt{x_2^2+(h_ic_i)^2} \tag{118}$$

where $c_i = y_p$ holds. At the time instant of coupling switching, additional kinetic energy is generated as initial velocity and x_2 is changed in this time instant ($x_1(t) = c_i = y_p$). From the point of view of the qualitative dynamic system theory, an additional increment of velocity in the initial conditions gives an instantaneous impulse on eqn (116) with a pulse height proportional to Δx_2 in the time instant of coupling switching. The model [eqn (116)] also accounts a possible change of dissipative forces in the time instant of coupling switching.

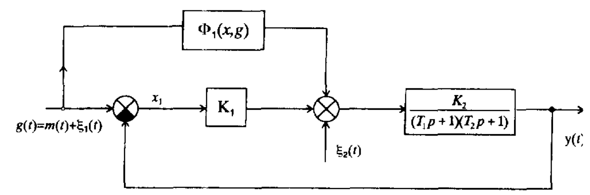

Fig. 9. Block diagram of dynamic system with time-variant characteristics.

In order to determine the probability density function $P(x,t)$, it is necessary to search for the solutions of eqns (110) and (111) with the boundary conditions of eqns (112) and (115b), and with natural normalization condition as:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}P(x_1,x_2,t)\,\mathrm{d}x_1\,\mathrm{d}x_2=1 \tag{119}$$

The stationary probability density function $P(x_1, x_2)$ is the solution of the FPK eqns (110) and (111) for $\partial P/\partial t = 0$:

$$P(x_1,x_2)\begin{cases}C\exp\left\{-\dfrac{2\alpha}{C_f^2}[x_2^2+\omega_0^2x_1^2]\right\}, \text{ until } |x_1|<c_0\\ C\exp\left\{-\dfrac{2\alpha}{C_f^2}[x_2^2-\Delta x_2(2x_2^{+}+\Delta x_2)\right.\\ \left.\qquad +\omega_i^2x_1^2+h_ic_i^2]\right\},\\ \text{for } |x_1|\geq c_i \vee |x_1|<c_{i+1}\ (i=0,\ 1,\ldots,n)\end{cases} \tag{120}$$

Now we present $P(x, t)$ (inside domains $\{I\}$ and $\{II\}$ in Fig. 7 separated by $x_1 = c_0$) in the form of integrals from P_1 and P_2 in eqns (112) and (115b) (see Appendix B). In domain $\{I\}$ ($0 \leq x_1 < c_0$; $0 \leq (x_2 = \xi) < \infty$):

$$\begin{aligned}P(x_1,x_2,t)=&\int_0^{\infty}\mathrm{d}\xi\int_0^{c_0}f(\eta,\xi)q_1(\tau=0)\,\mathrm{d}\eta\\ &-\int_0^t\mathrm{d}\tau\int_0^{\infty}[\xi P_1(c_0,\xi,\tau)q_1(c_0,\xi,\tau)\\ &-\xi P_1(0,\xi,\tau)q_1(0,\xi,\tau)]\,\mathrm{d}\xi\\ &+\int_0^t\mathrm{d}\,\tau\int_0^{c_0}\left\{\left[\omega_0^2\eta q_1P_1(\eta,\tau)\right.\right.\\ &\left.-\frac{C_f^2}{2}P_2(\eta,\tau)\right]q_1(\xi=0)\\ &\left.+\frac{C_f^2}{2}P_1(\eta,\tau)\frac{\partial q_1}{\partial\xi}(\xi=0)\right\}\mathrm{d}\eta\end{aligned} \tag{121}$$

and in domain $\{II\}$ ($c_0 \leq x_1 < c_1$)

$$\begin{aligned}P(x_1,x_2,t)=&\int_0^{\infty}\mathrm{d}\xi\int_0^{c_1}f(\eta,\xi)q_2(\tau=0)\,\mathrm{d}\eta\\ &-\int_0^t\mathrm{d}\tau\int_0^{\infty}[\xi P_1(c_1,\xi,\tau)q_2(c_1,\xi,\tau)\\ &-\xi P_1(c_0,\xi,\tau)q_2(0,\xi,\tau)]\,\mathrm{d}\xi\\ &+\int_0^t\mathrm{d}\tau\int_{c_0}^{c_1}\left\{\left[\left(\omega_1^2\eta-2\alpha\frac{c_0^2h_1}{2x_2^{+}}+h_1c_0\right)\right.\right.\end{aligned}$$

$$\times P_1(\eta,\tau)-\frac{C_f^2}{2}P_2(\eta,\tau)\Bigg]q_2(\xi=0)$$

$$+\frac{C_f^2}{2}P_1(\eta,\tau)\frac{\partial q_2}{\partial \xi}(\xi=0)\Bigg\}\,\mathrm{d}\eta \qquad (122)$$

where $P(x_1,x_2,0)=f(x_1,\ x_2)$, and $q_1(x_1,x_2,t,\xi,\tau)$, and $q_2(x_1,\ x_2,\ t,\ \xi,\tau)$ are probability density functions for a Markovian process satisfying eqn (108) under the conditions $x\in\{I\}$ and $x\in\{II\}$, respectively (see Appendix B).

Using boundary the conditions of eqns (112) and (115), as well as eqns (121) and (122), the boundary functions $P_1(x_2, t)$ and $P_2(x_2, t)$ are determined as:

$$P_1(x_2,t)=\int_0^\infty \mathrm{d}\xi\int_{c_0}^{c_1} f(\eta,\xi)q_2(x_1=c_0,\tau=0)\,\mathrm{d}\eta$$

$$+\int_0^t \mathrm{d}\tau\int_0^\infty \xi P_1(c_0,\xi,\tau)q_2(x_1=c_0,\eta=c_0)\,\mathrm{d}\xi$$

$$+\int_0^t \mathrm{d}\tau\int_{c_0}^{c_1}\Bigg\{\Bigg[(\omega_0^2\eta-2\alpha\frac{c_0^2h_1}{2x_2^+}+h_1c_0)P_1(\eta,\tau)$$

$$-\frac{C_f^2}{2}P_2(\eta,\tau)\Bigg]q_2(x_1=c_0,\xi=0)$$

$$+\frac{C_f^2}{2}P_1(\eta,\tau)\frac{\partial q_2}{\partial \xi}(x_1=c_0,\xi=0)\Bigg\}\,\mathrm{d}\eta \qquad (123)$$

and

$$P_2(x_2,t)=\int_0^\infty \mathrm{d}\xi\int_0^{c_0} f(\eta,\xi)\frac{\partial q_1}{\partial x_2}(x_1=c_0,\tau=0)\,\mathrm{d}\eta$$

$$-\int_0^t \mathrm{d}\tau\int_0^\infty\Bigg[\xi P_1(c_0,\xi,\tau)\frac{\partial q_1}{\partial x_2}(x_1=c_0,\eta=0)\Bigg]$$

$$\times\,\mathrm{d}\xi+\int_0^t \mathrm{d}\tau\int_0^{c_0}\Bigg\{\Bigg[\omega_0^2\eta P_1(\eta,\tau)$$

$$-\frac{C_f^2}{2}P_2(\eta,\tau)\frac{\partial q_1}{\partial x_2}(x_1=c_0,\xi=0)\Bigg]$$

$$+\frac{C_f^2}{2}P_1(\eta,\tau)\frac{\partial^2 q_1}{\partial x_2\partial\xi}(x_1=c_0,\xi=0)\Bigg\}\mathrm{d}\eta \qquad (124)$$

The solutions of eqns (123) and (124) as P_1 and P_2 are discussed in Appendix B. For P_1 and P_2, it is possible to find the probability density function $P(x, t)$ from eqns (121) and (122) in domains $\{I\}$ and $\{II\}$, respectively.

Eqns (123) and (124) become solutions in the domains $\{III\}$ and $\{IV\}$ (see Fig. 7) as long as the integration limits for η as change from c_i to $-\ c_i$ and for ξ to $(-\infty, 0]$.

Special Case. Consider the model of a rod system with additional elastic coupling as in Part 1 of this paper:[1]

$$\ddot{x}+[2\beta_0+2\mu\alpha(t)]\dot{x}+\Omega^2[1+2\sigma\chi(t)]x$$

$$+2\kappa(\ddot{x}x+\dot{x}^2)x+\gamma_1x^3+\gamma_2x^5+\varepsilon\dot{x}x^2=\nu\eta(t) \qquad (125)$$

In the time instant when random vibration of this system achieves or acrosses any level of amplitude A_0, the coupling get torn and, as an example, one parameter, γ_1, changes as an irreversible jump. Additional coupling is defined as in Part 1[1] accounting for coefficients γ_1, I_2 and I_3 from the paper.[1] Determination of the probability density function is very important for the safety of a similar system described by eqn (125).

In this case, the hyperplane of the form of eqn (103) for phase space $(A, \dot{A})$ of eqn (125) is described as $|A| = A_0$ and a jump change of parameter γ_1 is defined as:

$$\gamma_1=\gamma_{11}\ \text{until}\ |A|=A_0;\gamma_1=\gamma_{12},\ \text{for}\ |A|\geq A_0 \qquad (126)$$

The FPK equation in this case is found in Part 1 of this paper:[1]

$$\frac{\partial w(A,t)}{\partial t}=\frac{\partial}{\partial A}\Bigg[\frac{a_7}{2}A^{-1}-\left(a_1-\frac{3a_8}{2}-a_9-a_{10}\right)A$$

$$-I_1A^3+I_2^+A^5+I_3^+A^7\Bigg]w(A,t)$$

$$+\frac{\partial^2}{2\partial A^2}[(a_7+a_8^*A^2)w(A,t)],\ \text{until}\ |A|<A_0 \qquad (127)$$

For $|A|\geq A_0$, parameters I_2^+ and I_3^+ change to I_2^- and I_3^- and the FPK equation becomes:

$$\frac{\partial w(A,t)}{\partial t}=\frac{\partial}{\partial A}\Bigg[\frac{a_7}{2}A^{-1}-\left(a_1-\frac{3a_8}{2}-a_9-a_{10}\right)A$$

$$-I_1A^3+I_2^-A^5+I_3^-A^7\Bigg]w(A,t)$$

$$+\frac{1}{2}\frac{\partial^2}{\partial A^2}[(a_7+a_8^*A^2)w(A,t)],\ \text{for}\ |A|\geq A_0 \qquad (128)$$

where $I_2^+=\gamma_{11}\varepsilon/32\Omega^2$; $I_3^+=5\gamma_{11}\varepsilon/256\Omega^2$; $I_2^-=\gamma_{12}\varepsilon/32\Omega^2$; and $I_3^-=5\gamma_{12}\varepsilon/256\Omega^2$. With condition $|A| = A_0$, the phase space $(A, \dot{A})$ of eqn (125) is divided into two domains. We consider the first change of the system characteristics, as an example, in the domain $(0\leq A\leq A_0,\ 0\leq\dot{A}<\infty)$. Since the diffusion coefficient B of eqn (125) is continuous on switching line $A = A_0$, the first condition of the conjugation eqn (106) can be written as:

$$w^+=w^-=w_1(A_0,t) \qquad (129)$$

where w^+ is the limit value of probability density function $w(A, t)$ as A approaches the switching line in the domain $A<A_0$; w^- is the corresponding limit value as A approaches to the switching line in domain $A\geq A_0$.

The second condition of conjugation eqn (107) is defined as the subtraction of eqn (127) from eqn (128):

$$\frac{\partial w^+(A,t)}{\partial t}=\frac{\partial w^-(A,t)}{\partial t}$$

$$+2[(I_2^+-I_2^-)A_0^5+(I_3^+-I_3^-)A_0^7]w_1$$

$$\times(a_7+a_8^*A^2)^{-1} \qquad (130)$$

where

$$\frac{\partial w^{+}(A,t)}{\partial t}=\left.\frac{\partial w^{+}(A,t)}{\partial t}\right|_{A=A_0+0};$$

$$\frac{\partial w^{-}(A,t)}{\partial t}=\left.\frac{\partial w^{-}(A,t)}{\partial t}\right|_{A=A_0-0} \tag{130a}$$

In order to determine the probability density function $w(A,t)$, it is necessary to find solutions of eqns (127) and (128)) with the boundary conditions of eqns (129) and (130). The stationary and nonstationary solutions of eqns (127) and (128) are found from corresponding equations in Part 1.[1] In this case, the approach suggested in Part 1[1] helps to get rid of complex calculations of mixed integral equations.

5.2 Case 2: stochastic processes in time-variant systems with sliding noise mode

Consider one specific case of the dynamic system of eqn (108) as:

$$\dot{x}_1=x_2;\ \dot{x}_2=-2\alpha x_2-\omega_0^2 x_1+\ddot{g}(t)+2\alpha\dot{g}(t)+\Phi(x,g)g(t) \tag{131}$$

The stochastic process $f(t)$ in eqn (108) is now described as output $g(t)$ of a shaping filter with time-variant characteristics and $\omega^2(x_1,t)=\omega_0^2=constant$. Eqn (131) represents a control process of equipment vibration isolation using combination of a servo system and a sliding mode. Fig. 9 shows the block diagram of servo system where:

$$2\alpha=\frac{T_1+T_2}{T_1T_2},\ \omega_0^2=\frac{1+k_1k_2}{T_1T_2},\ \Phi(x,g)=\frac{1-\varphi k_2}{T_1T_2},$$

$$\varphi(x,g)=\begin{cases}-l, & \text{if } (c_1x_1+c_2x_2)g(t)>0\\ +l, & \text{if } (c_1x_1+c_2x_2)g(t)\le 0\end{cases} \tag{131a}$$

In reality, $lk_2 \gg 1$ and:

$$\Phi(x,g)=\begin{cases}-k, & \text{if } (c_1x_1+c_2x_2)g(t)>0\\ +k, & \text{if } (c_1x_1+c_2x_2)g(t)\le 0,\ k=lk_2/T_1T_2\end{cases} \tag{131b}$$

Analysis of systems of this type shows[17,24] that the output x_1 of such a system turns out to be invariant with respect to a broad class of the reproducible process $g(t)$. Consider the case of random vibration excitation $\xi_2(t)$ in input $g(t)$ as additive component white noise with $M[\xi_2(t)]=0$ and $M[\xi_2(t)\xi_2(t+\tau)]=a\delta(\tau)$. The equation of motion [eqn (131)] assumes the forms:

$$\dot{x}_1=x_2;\ \dot{x}_2=-2\alpha x_2-\omega_0^2x_1+\Phi_0(x,g)+\frac{k_2}{T_1T_2}\xi_2 \tag{132a}$$

$$\Phi_0(x,g)=\begin{cases}\Phi_1=\ddot{g}(t)+2\alpha\dot{g}(t)-kg(t),\\ \quad \text{if } (c_1x_1+c_2x_2)g(t)>0\\ \Phi_2=\ddot{g}(t)+2\alpha\dot{g}(t)-kg(t),\\ \quad \text{if } (c_1x_1+c_2x_2)g(t)\le 0\end{cases} \tag{132b}$$

The FPK equations for describing stochastic sensitivity of vibration isolation to the random excitation in this case assume the forms:

$$\begin{aligned}L[P(x_1,x_2,t)]=&-\frac{\partial P}{\partial t}-\frac{\partial}{\partial x_1}\{x_2P\}\\&-\frac{\partial}{\partial x_1}\{[-2\alpha x_2-\omega_0^2x_1+\Phi_1]P\}\\&+\frac{C_f^2}{2}\frac{\partial^2P}{\partial x_2^2}=0,\ \text{if } (c_1x_1+c_2x_2)g(t)>0\end{aligned} \tag{133}$$

$$\begin{aligned}L[P(x_1,x_2,t)]=&-\frac{\partial P}{\partial t}-\frac{\partial}{\partial x_1}\{x_2P\}\\&-\frac{\partial}{\partial x_1}\{[-2\alpha x_2-\omega_0^2x_1+\Phi_2]P\}\\&+\frac{C_f^2}{2}\frac{\partial^2P}{\partial x_2^2}=0,\ \text{if } (c_1x_1+c_2x_2)g(t)\le 0\end{aligned} \tag{134}$$

Without limiting generality, assume that $g(t)>0$ and the problem can be solved in the system space $x=(x_1,x_2)$, which is broken into parts by the sliding mode condition as $(c_1x_1+c_2x_2)=0$: $\{I\}=\{x:(c_1x_1+c_2x_2)>0\}$ and $\{II\}=\{x:(c_1x_1+c_2x_2)<0\}$.

As in Section 5.1, the diffusion coefficient B_{ij} is continuous on the switch line [eqn (103)] and the matching condition [eqn (106)] assumes the form:

$$P^{+}=P^{-}=P_1(x_2,t) \tag{135}$$

where, as in Section 5.1, P^{+} is the limiting value of probability density function $P(x,t)$ as x approaches the line $x:(c_1x_1+c_2x_2)=0$ in domain $\{I\}$; P^{-} is the corresponding limiting value in domain $\{II\}$.

The condition eqn (107), in this case, becomes:

$$\left(\frac{\partial P}{\partial x_2}\right)^{-}=P_2+\frac{4k}{C_f^2}P_1g,P_2(x_2,t)=\left(\frac{\partial P}{\partial x_2}\right)^{+} \tag{136}$$

For an important special case where $g(t)=g_1=constant$ and the switching line becomes $x_1=0$ (the stability problem of vibration compensation), it is possible to find the stationary probability density function $P(x_1,x_2)$ as the solution of the FPK equation system. Setting $\partial P/\partial t = 0$, we obtain as in Section 5.1:

$$P(x_1,x_2) = \begin{cases} C\exp\left\{-\dfrac{2\alpha}{C_f^2}\left[x_2^2+\omega_0^2\left(x_1+\dfrac{kg_1}{\omega_0^2}\right)^2\right]\right\}, \\ \text{if } x_1 > 0 \\ C\exp\left\{-\dfrac{2\alpha}{C_f^2}\left[x_2^2+[x_2^2+\omega_0^2\left(x_1-\dfrac{kg_1}{\omega_0^2}\right)^2\right]\right\}, \\ \text{if } x_1 \le 0 \end{cases} \tag{137}$$

The constant C is determined from the normalization condition.

It is required to find the nonstationary solution for an arbitrary input function $g(t)$ and switching line [eqn (103)]. As in Section 5.1, we present $P(x, t)$ inside the domains $\{I\}$ and $\{II\}$ separated by the condition $c_1x_1 + c_2x_2 = 0$, $c_1 > 0$ and $c_2 > 0$ in the form of integrals of the boudary functions P_1 and P_2.

For domain $\{I\}$, according of Appendix B as in Section 5.1, one obtains:

$$\begin{aligned} P(x_1,x_2,t) = & \int_{-\infty}^{\infty} \mathrm{d}\xi \int_{-c_2/c_1\eta}^{\infty} f(\eta,\xi) q_1(\tau=0)\,\mathrm{d}\eta \\ & + \int_0^t \mathrm{d}\tau \int_{-\infty}^{\infty} \xi P_1(\xi,\tau) q_1\left(\eta = -\frac{c_2}{c_1}\xi\right)\mathrm{d}\xi \\ & + \int_0^t \mathrm{d}\tau \int_{-\infty}^{\infty} \left\{\left[\omega_0^2\eta - 2\alpha\frac{c_1}{c_2}\eta \right.\right. \\ & \left. - \Phi_1(\tau))P_1(\eta,\tau) \right. \\ & \left. - \frac{C_f^2}{2}P_2(\eta,\tau)\right] q_1\left(\xi = -\frac{c_1}{c_2}\eta\right) \\ & \left. + \frac{C_f^2}{2}P_1(\eta,\tau)\frac{\partial q_1}{\partial \xi}\left(\xi = -\frac{c_1}{c_2}\eta\right)\right\}\mathrm{d}\eta \end{aligned} \tag{138}$$

For domain $\{II\}$, taking into account the boundary conditions of eqns (135) and (136), one finds (see Appendix B):

$$\begin{aligned} P(x_1,x_2,t) = & \int_{-\infty}^{\infty} \mathrm{d}\xi \int_{-\infty}^{-c_2/c_1\xi} f(\eta,\xi) q_2(\tau=0)\,\mathrm{d}\eta \\ & - \int_0^t \mathrm{d}\tau \int_{-\infty}^{\infty} \xi P_1(\xi,\tau) q_2\left(\eta = -\frac{c_2}{c_1}\xi\right)\mathrm{d}\xi \\ & - \int_0^t \mathrm{d}\tau \int_{-\infty}^{\infty} \left\{\left[\omega_0^2\eta - 2\alpha\frac{c_1}{c_2}\eta - \Phi_2(\tau)\right.\right. \\ & - 2kg(\tau))P_1(\eta,\tau) \\ & \left. - \frac{C_f^2}{2}P_2(\eta,\tau)\right] q_2\left(\xi = -\frac{c_1}{c_2}\eta\right) \\ & \left. + \frac{C_f^2}{2}P_1(\eta,\tau)\frac{\partial q_2}{\partial \xi}\left(\xi = -\frac{c_1}{c_2}\eta\right)\right\}\mathrm{d}\eta \end{aligned} \tag{139}$$

According to Appendix B, $P(x_1, x_2, 0) = f(x_1, x_2)$, $q_1(x_1, x_2, t, \xi, \tau)$, and $q_2(x_1, x_2, t, \xi, \tau)$ are probability density functions for a Markovian process satisfying the eqn (131) under the conditions $x \in \{I\}$ and $x \in \{II\}$, respectively (see Appendix B). As in Section 5.1, using the boundary conditions of eqns (135) and (136) for determining the boundary functions we obtain, from eqns (138) and (139), the following linear integral equations:

$$\begin{aligned} P_1(x_2,t) = & \int_{-\infty}^{\infty} \mathrm{d}\xi \int_{-\infty}^{-c_2/c_1\xi} f(\eta,\xi) \\ & \times q_2\left(x_1 = -\frac{c_2}{c_1}x_2, \tau = 0\right)\mathrm{d}\eta \\ & - \int_0^t \mathrm{d}\tau \int_{-\infty}^{\infty} \xi P_1(\xi,\tau) \\ & \times q_2\left(x_1 = -\frac{c_2}{c_1}x_2, \eta = -\frac{c_2}{c_1}\xi\right)\mathrm{d}\xi \\ & - \int_0^t \mathrm{d}\tau \int_{-\infty}^{\infty} \left\{\left[\omega_0^2\eta - 2\alpha\frac{c_1}{c_2}\eta - \Phi_2(\tau)\right.\right. \\ & \left. - 2kg(\tau))P_1(\eta,\tau) - \frac{C_f^2}{2}P_2(\eta,\tau)\right] \\ & \times q_2\left(x_1 = -\frac{c_2}{c_1}x_2, \xi = -\frac{c_1}{c_2}\eta\right) \\ & \left. + \frac{C_f^2}{2}P_1(\eta,\tau)\frac{\partial q_2}{\partial \xi}\left(x_1 = -\frac{c_2}{c_1}x_2, \xi = -\frac{c_1}{c_2}\eta\right)\right\} \\ & \times \mathrm{d}\eta \end{aligned} \tag{140}$$

and

$$\begin{aligned} P_2(x_2,t) = & \int_{-\infty}^{\infty} \mathrm{d}\xi \int_{-\frac{c_2}{c_1}\xi}^{\infty} f(\eta,\xi) \\ & \times \frac{\partial q_1}{\partial x_2}\left(x_1 = -\frac{c_2}{c_1}x_2, \tau = 0\right)\mathrm{d}\eta \\ & + \int_0^t \mathrm{d}\tau \int_{-\infty}^{\infty} \xi P_1(\xi,\tau) \\ & \times \frac{\partial q_1}{\partial x_2}\left(x_1 = -\frac{c_2}{c_1}x_2, \eta = -\frac{c_2}{c_1}\xi\right)\mathrm{d}\xi \\ & + \int_0^t \mathrm{d}\tau \int_{-\infty}^{\infty} \left\{\left[\left(\omega_0^2\eta - 2\alpha\frac{c_1}{c_2}\eta - \Phi_1(\tau)\right)\right.\right. \\ & \left. \times P_1(\eta,\tau) - \frac{C_f^2}{2}P_2(\eta,\tau)\right] \\ & \times \frac{\partial q_1}{\partial x_2}\left(x_1 = -\frac{c_2}{c_1}x_2, \xi = -\frac{c_1}{c_2}\eta\right) \\ & \left. + \frac{C_f^2}{2}P_1(\eta,\tau)\frac{\partial^2 q_1}{\partial x_2 \partial \xi}\left(x_1 = -\frac{c_2}{c_1}x_2, \xi = -\frac{c_1}{c_2}\eta\right)\right\} \\ & \times \mathrm{d}\eta \end{aligned} \tag{141}$$

For engineering analysis of stochastic process in time-variant dynamic systems, method of statistical linearization is also effective[17,24].

6 CONCLUSIONS

In this paper, the probabilistic description of the response of a nonlinear time-variant dynamic system driven by external random processes has been discussed. The applications of the FPK method to stochastic analysis of nonlinear systems with random time-variant (reversible and irreversible) characteristics are considered. A new class of dynamic systems with stochastic nonlinearity and jump parametric excitations are introduced. Comparison of different statistical methods such as statistical linearization and statistical moments is performed. On the basis of the FPK equation, a new approximating method of statistical analysis for this class of nonlinear parametric systems is introduced. Using the solution of the FPK equation, the transient and stationary processes of these systems were studied. The sensitivity of the nonlinear system to different stochastic correlated parametric and jump excitations is studied. It is shown that the dynamic effect of stochastic nonlinearity on the response may be significant and the peak amplitude of the nonstationary moment response may be less than that of the linear stochastic parametric case. Moreover the third statistical moment can be positive and negative for the essential nonlinear dynamic systems. In this second part of the paper, we provide a background for studying the stochastic stability of a nonlinear dynamic system with stochastic nonlinearity and stochastic time-variant characteristics. This study is to be described in Part 3 of this paper.

REFERENCES

1. Ulyanov, S.V., Feng, M., Ulyanov, V.S., Yamafuji, K., Fukuda, T. and Arai, F. Stochastic analysis of time variant nonlinear dynamic systems. Part 1: The Fokker–Planck–Kolmogorov equation approach. *Probabilistic Engineering Mechanics*, 1998, **13**, 183–203.
2. Nikolaenko, N. A. and Ulyanov, S. V., *Statistical Dynamic of Machine-Building Constructions*. Mashinostroenie, Moscow, 1977 (in Russian).
3. Goldenblat, I. I., Nikolaenko, N. A. and Ulyanov, S. V., *Models of Aseismic Constructions*. Nauka, Moscow, 1979 (in Russian).
4. Bolotin, V. V., *Random Vibration of Elastic Systems*. Nauka, Moscow, 1979 (in Russian).
5. Roberts, J. B. and Spanos, P. D., *Random Vibration and Statistical Linearization*. Wiley, NY, 1990.
6. Di Paola, M., Ricciardi, G. and Vasta, M. A method for the probabilistic analysis of nonlinear systems. *Probabilistic Engineering Mechanics*, 1995, **10**, 1–10.
7. Malakhov, A. N., *Cumulant Analysis of Stochastic Non-Gaussian Processes and Its Transformations*. Sovetskoe Radio, Moscow, 1978 (in Russian).
8. Wang, R., Kusumoto, S. and Zhang, Z. A new equivalent non-linearization technique. *Probabilistic Engineering Mechanics*, 1996, **11**, 129–137.
9. Polidori, D. C. and Beck, J. L. Approximate solutions for non-linear random vibration problems. *Probabilistic Engineering Mechanics*, 1996, **11**, 179–185.
10. Donley, M. G. and Spanos, P. D., Dynamic analysis of non-linear structures by the method of statistical quadratization. In *Lecture Notes in Engineering*. Springer, Berlin, 1990.
11. Eu, B. C. and Bhattacharya, D. K. Modified-moment method for the Fokker-Planck equation and some aspects of the thermodynamics of irreversible processes. *Physical Review*, 1989, **39A**, 3597–3608.
12. Sigalov, G. G. To analysing nonlinear systems with the aid of moment equations. *Avtomatika i Telemekhanika*, 1970, **6**, 37–47. (in Russian).
13. Evlanov, L. G. and Konstantinov, V. M., *The System with Random Parameters*. Nauka, Moscow, 1976 (in Russian).
14. Andrews, L. C. The probability density function for the output of a cross-correlator with bandpass inputs. *IEEE Transactions on Information Theory*, 1973, **19**, 13–19.
15. Miller, L. E. and Lee, J. S. The probability density function for the output of an analog cross-correlator with bandpass inputs. *IEEE Transactions on Information Theory*, 1974, **20**, 933–940.
16. Dimentberg, M. F., *Statistical Dynamics of Nonlinear and Time-Varying Systems*. Research Studies Press, Letchworth, UK, 1988.
17. Petrov, B. N., Ulanov, G. M., Ulyanov, S. V. and Khazen, E. M., *Information and Semantic Problems in Control Processes and Organization*. Nauka, Moscow, 1977 (in Russian); Ulanov, G. M. and Ulyanov, S. V., Dynamic systems with fuzzy and stochastic variable structures (stochastic vibrations, coherent states and solitons in classical, relativistic and quantum control systems). *Engineering Cybernetics*, 1982, **15**, 3–145 (in Russian).
18. Matveev, P. S., Glebachev, Yu. M. and Sinitsyn, A. S. Research of stochastic systems with methods of invariant and sensitivity theories. *Proceedings of the Conference on Sensitivity and Invariant Theories in Control Systems*, 1976, **III**, 239–251. (in Russian).
19. Sklyarevich, A. N., *Introduction in Statistical Dynamic of Systems with Possible Violations*. Zinatne, Riga, 1973 (in Russian).
20. Repin, V. G. Analysis of one class of systems with randomly varying parameters. *Avtomatika i Telemekhanika*, 1970, **6**, 21–28. (in Russian).
21. Stratonovich, R. L. The Fokker-Planck equation with discontinuous coefficients and conditions on the surface of discontinuity. *Izvestiya Vysshich Uchebn. Zaved. Ser. Radiofizika*, 1965, **8**, 861–871. (in Russian).
22. Tai, M. L., Matching conditions for the transition density function for a multidimensional Markov process. *Izvestiya Vysshich Uchebn. Zaved. Ser. Radiofizika*, 1965, **8**, 237–246 (in Russian); Markov processes with discontinuous drift and diffusion coefficients. *Izvestiya Vysshich Uchebn. Zaved. Ser. Radiofizika*, 1966, **9**, 834–841 (in Russian).
23. Stratonovich, R. L., *Topics in the Theory of Random Noise*. Vol. 1. Gordon and Breach, NY, 1963.

24. Nikitin, A. K. and Ulanov, G. M., Application of the Kolmogorov equations to the analysis of stochastic processes in variable-structure system. *Avtomatika i Telemekhanika*, 1968, **4**, 34–41 (in Russian); Investigating random processes in control systems with a variable structure using linearization methods. *Avtomatika i Telemekhanika*, 1968, **6**, 50–62 (in Russian).
25. Eshov, I. I. and Skorochod, A. V., Markovian processes homogeneous on second component. Parts I and II. *Probability Theory and its Applications*, 1969, **14**, 3–23 and 481–496 (in Russian).
26. Kazakov, I. E. and Artemiev, V. M., *Optimization of Dynamical Systems with Random Structure.* Nauka, Moscow, 1980 (in Russian).
27. Kazakov, I. E., *Statistical Theory of Control System in State Space.* Nauka, Moscow, 1975 (in Russian).

APPENDIX A STOCHASTIC JUMP IN SYSTEM CHARACTERISTICS

Appendix A.1 Reversible stochastic jump in system characteristics

Consider any Markovian stochastic process $\xi(t)$ with a smooth non-stop set Ω of states and with a real value-reversible jumps. For any Markovian stochastic process with a non-stop set of states:

$$P_s^t(\omega|\omega'') = \int_\Omega P_u^t(\omega|\omega')P_s^u(\omega'|\omega'')\,\mathrm{d}\omega' \qquad \text{(A1)}$$

If a jump in a small time interval τ has the probability $a\tau + 0(\tau)$ and the probabity density function of the jump as probability of jump from point x into interval $[y, y + \mathrm{d}y]$ is $f(x, y)\,\mathrm{d}y + 0(y)$ and $\int_{-\infty}^{\infty} f(x,y)\,\mathrm{d}y = 1$, then:

$$P_\tau(x|x) = 1 - a\tau + 0(\tau);\; P_\tau(y + \mathrm{d}y|x) = a\tau f(x,y)\,\mathrm{d}y \qquad \text{(A2)}$$

For the probability density function $w(y, t)$, one can write:

$$w(y, t + \Delta t) = (1 - a\Delta t)w(y,t) + \int_{-\infty}^{\infty} a\Delta t w(x,t)f(x,y)\,\mathrm{d}x \qquad \text{(A3)}$$

For the probability $w(y,t)\,\mathrm{d}y$ that point $x(t)$ at the time instant t is in the interval $[y, y + \mathrm{d}y]$, one obtains from eqn (A3):

$$\frac{\partial}{\partial t}w(y,t) = -aw(y,t) + \int_{-\infty}^{\infty} aw(x,t)f(x,y)\,\mathrm{d}x \qquad \text{(A4)}$$

eqn (A4) is the integral-differential equation. The availability of the integral term is typical for a stochastic process with jumps. On the basis of eqns (A2) and (A3), and Ref.[20]:

$$\begin{aligned} w^{(a)}(x, t+\Delta t) &= (1 - \nu^{(aa)}\Delta t)\int w_1^{(a)}(x, t+\Delta t; y, t)w^{(a)} \\ &\times (y,t)\,\mathrm{d}y + \sum_{b\neq a}\int_t^{t+\Delta t} \nu^{(ab)}(\tau)\,\mathrm{d}\tau \\ &\int w_1^{(a)}(x, t+\Delta t; y, \tau)\,\mathrm{d}y \\ &\times \int w_1^{(b)}(y, \tau; \eta, t)w^{(b)}(\eta, t)\,\mathrm{d}\eta \end{aligned} \qquad \text{(A5)}$$

eqn (A5) suggests that for the time interval $(t, t + \Delta t)$, only one jump is possible. In eqn (A5) the stochastic process $\varphi_{kl}(x, t)$ from eqn (80) is the Poisson process with real up–down jumps and integral terms including possible states of the stochastic process $\varphi_{kl}(x, t)$. For a small value of $(t - \tau)$, the following relationship for the probability density function $w_i^{(a)}(x, t; y, \tau)$ can be obtained:[17,20]

$$\int w_1^{(a)}(x, t; y, \tau)f(y)\,\mathrm{d}y = f(x) + L^{(a)}f(x)(t - \tau) + 0(t - \tau) \qquad \text{(A6)}$$

where

$$L^{(a)} = -\sum_{k=1}^{n}\frac{\partial}{\partial x_k}A_k^{(a)} + \frac{1}{2}\sum_{k,l=1}^{n}\frac{\partial^2}{\partial x_k\partial x_l}B_{kl}^{(a)} \qquad \text{(A7)}$$

Substituting eqn (A6) into eqn (A5):

$$\begin{aligned} w^{(a)}(x, t+\Delta t) &= (1 - \nu^{(aa)}\Delta t)(w^{(a)}(x,t) + L^{(a)}w^{(a)}(x,t)\Delta t \\ &+ O(\Delta t)) + \sum_{b\neq a}\nu^{(ab)}\Delta t(w^{(b)}(x,t) + 0(\Delta t)) \end{aligned} \qquad \text{(A8)}$$

Reform eqn (A8) as:

$$\begin{aligned} &\frac{w^{(a)}(x, t+\Delta t) - w^{(a)}(x,t)}{\Delta t} \\ &= L^{(a)}w^{(a)}(x,t) - \nu^{(aa)}w^{(a)}(x,t) \\ &+ \sum_{b\neq a}\nu^{(ab)}(w^{(b)}(x,t) + 0(\Delta t)) \end{aligned} \qquad \text{(A9)}$$

From eqn (A9) for $\Delta t \to 0$ eqn (84) is obtained.

Appendix A.2 Irreversible stochastic jump in system characteristics

From a physical point of view, the stochastic process $\varphi_{kl}(x,t)$ in eqn (80) is a random vector process described as a random irreversible jump process with negative values. For a fixed value of first component $x = c_i$, the stochastic process $\varphi_{kl}(x, t)$ is a random homogeneous additive jump process with conditional independent increments on the second component t. For an evaluation of the conditional probability density function of the stochastic process $\varphi_{kl}(x,t)$ on the second variable t with the first variable $x(c_i)$ fixed it is sufficient[25] to find the characteristic function on the second variable.[2,25] Therefore, the stochastic process $\varphi_{kl}(x, t)$ can be described as conditionally independent of two components in the temporal domain or in the phase plane as:

$$\varphi(x,t) = \begin{cases} \varphi_1(t), |x| = c_i \\ \varphi_{2i}(x) \end{cases} \qquad \text{(A10)}$$

Consider here the first case of eqn (A10). The second case of eqn (A10), describing a stochastic system with timevariant characteristics a sliding mode, is discussed for eqn (131). Conditional characteristic function of process $\varphi_l(t)$ is described as:[25]

$$M[\exp\{i\lambda\varphi_1(x,t)\}] = \exp\{t\psi\} \quad (A11)$$

where the cumulant function

$$\psi(\lambda) = i\lambda h + \int_{-N}^{\infty}\{\exp(i\lambda y) - 1\}\,\mathrm{d}F(y) \quad (A12)$$

$h = \{h_1, h_2, \ldots, h_n\}$, $N = \{N_1, N_2, \ldots, N_n\}$, $h_i \in \mathrm{N}$ are vectors of random variables characterized the 'depth' of the jump $F(h < y)$ is the distribution function of a random jump value in process $\varphi_1(t)$.

Consider a general case in which stochastic process $\{x(t);\ \varphi_{kl}(x,t)\}$ is the Markovian process with conditionally independent homogenous second component at a fixed value of the first component.[2,25] Consider $T[t,\tau,x_i]$ as the set of sampling trajectories of process $x(t)$ with $|x| \le x_i$ for $t \le u \le \tau$. The probability of this set for any initial state $x(t_0)$ is positive. Let $\alpha_0^i(\varphi_{kl})$ be the conditional characteristic function of process φ_{kl} with a fixed first component x_i and consider $\alpha_0^i(\varphi_{kl})$ along a trajectory $x(t)$ from the set $T[t,\tau,x_i]$. Examine the crossing from state x_{i-1} to state x_i. The process $x(t)$ is a stochastically continuous Markovian process. If the initial state $x(t_0) = x_o$ and $|x(t)| = x_1$, then the process φ_{kl} for state $|x(t)| < x_1$, has increments coincided with increments of any process $f_{x1}(t)$. Characteristic function of this process $f_{x1}(t)$ coincides with characteristic function $\alpha_0^u(\varphi_{kl})$. In any time moment τ_1, process $x(t)$ crosses in state $x_1 \le |x(t)| < x_2$ and process φ_{kl} has a jump $f(\tau_1, x_0, x_1)$ with the characteristic function $\psi(\tau_1, x_0, x_1)$. This function $\psi(\tau_1, x_0, x_1)$ is dependent on crossing the time moment τ_1, the states before and after the moment of crossing. The values of $f(\tau_1, x_0, x_1)$ for different τ_1, x_0, x_1 are independent from $f_{x1}(t)$ and $x(t)$. When process $x(t)$ is in state $|x(t)| < x_2$, the increments of process φ_{kl} coincide with increments of any process $f_{x2}(t)$, and in the time moment τ_2 of crossing $x(t)$ from state $|x(t)| < x_2$ into state $x_2 \le |x(t)| < x_3$, process φ_{kl} has a jump of $f(\tau_2, x_1, x_2)$ and so on. Similar processes are stochastic processes homogenuous on the second component and with jumped first component.[2,25] On examined trajectory one can write:[25]

$$\alpha_u^0(\varphi_{kl}) = \alpha_{x_0}(0, \tau_1, \varphi_{kl}) \prod_{\tau_k < t} \psi(\tau_k, x_{k-1}, x_k, \varphi_{kl}) \times \prod_{\tau_{k+1} < t} \alpha_{x_k}(\tau_k, \tau_{k+1}, \varphi_{kl})\alpha_{x_i}(\tau_i, t, \varphi_{kl}) \quad (A13)$$

where i is obtained from following relationship $\tau_i \le t < \tau_{i+l}$; and $\psi(\tau_k, x_{k-1}, x_k, \varphi_{kl}) = \lim_{\varepsilon \to 0} \alpha_{\tau_k - \varepsilon}(\varphi_{kl})$. The system with time-invariant characteristics is a special case of eqn (A13) where $f(\tau_i, x_i, x_{i+l})$, $\psi(\ldots) = 1$.

The probability $P^{(a)}(t)$ of the parameters of system being in state (a) is:

$$p^{(a)}(t) = \int_{-\infty}^{x} w^{(a)}(x,t)\,\mathrm{d}x \quad (A14)$$

which is evaluated from

$$\frac{\mathrm{d}P^{(a)}(t)}{\mathrm{d}t} = -\nu^{(aa)}p^{(a)}(t) + \sum_{b \ne a} \nu^{(ba)}P^{(b)}(x,t)\ (a = 1, 2, \ldots, r) \quad (A15)$$

with initial stateas $P^{(a)}(0) = P_0^{(a)}$ calculated from eqn (A14).

For the stochastic process $\varphi_1(t)$ in eqn (A10), the probability of absence from a change in characteristics shown in eqn (A15) is:

$$p^{(a)}(t) = \mathrm{e}^{-\nu^{(aa)}t} \quad (A16)$$

From eqns (A5) and (A16), the probability $w^{(a)}(x, t)$ for a system with irreversible characteristics[2,3,20] is found to be:

$$w(a)(x,t) = \mathrm{e}^{-\nu^{(aa)}t}\int w_1^{(a)}(x,t;y,0)w_0^{(a)}(y)\,\mathrm{d}y + \int_0^t \mathrm{e}^{-\nu^{(aa)}(t-\tau)}\,\mathrm{d}\tau \int w_1^{(a)}(x,t;y,\tau) \times \sum_{b=1}^{a-1} \nu^{(ab)}w^{(b)}(y,\tau)\,\mathrm{d}y \quad (A17)$$

The result of this approach differ from those described in Refs[19,26,27]

APPENDIX B MATCHING CONDITIONS FOR FPK EQUATIONS WITH DISCONTINUOUS COEFFICIENTS AND DYNAMIC SYSTEMS WITH VARIABLE STRUCTURE

The FPK eqn (104) may be interpreted in terms of field theory as a discontinuous equation. Introducing the equation:

$$\zeta_i = \{A_i, w\} - \sum_{j=1}^{n} \frac{1}{2}\frac{\partial^2}{\partial x_i}\{B_{ij}w\} \quad (B1)$$

where ζ_i is the probability flux density in the direction of the coordinate x_i, and we rewrite eqn (104) in the form

$$\frac{\partial w}{\partial t} = -\sum_{i=1}^{n}\frac{\partial \zeta_i}{\partial x_i} \equiv -div\zeta \quad (B2)$$

Let $c = (c_1, c_2, \ldots, c_n)^{\mathrm{T}}$ be the vector normal to the hyperplane [eqn (103)], and let υ be the coordinate along the normal $(\partial\upsilon)/(\partial x_i) = c_i$ and $\upsilon = 0$ on the hyperplane eqn (103).

Under the conditions of the problem, the flux density vector [eqn (B1)] can have a singularity of the type $\delta(\upsilon)$ on the hyperplane [eqn (103)], i.e. it is representable in the form:

$$\zeta_i = \zeta_i^o\delta(\nu) + \zeta_i' \quad (B3)$$

where ζ'_i is the discontinous part of the flux.

Designating $\Delta[f] = f(v = 0_+) - f(v = 0_-)$ we can write:

$$\zeta_i^0 = -\frac{1}{2}\sum_{j=1}^{n} c_i \Delta\{B_{ij} w\} \qquad \text{(B4)}$$

$$\zeta'_i = A_i w - \frac{1}{2}\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\{B_{ij} w\} \qquad \text{(B5)}$$

where the differentiation in the last term is carried out without regard for the sudden jump.

Under the reasonable assumption that $w(x, t)$ does not have δ-singularities, the matching conditions acquire the form:

$$\sum_{i,j=1}^{n} c_i c_j \Delta\{B_{ij} w\} = 0 \qquad \text{(B6)}$$

$$div_r \zeta^0 + \sum_{i=1}^{n} c_i \Delta[\zeta_i'] = 0 \qquad \text{(B7)}$$

Here, $div_r \zeta^0$ is the $(n - 1)$-dimensional divergence on the hyperplane [eqn (118)].

From eqns (B8) and (B7), we can write the matching conditions for FPK eqns (104), (110) and (111) as follows:

$$\Delta[B_{ij} \cdot P(x, t)] = 0 \qquad \text{(B8)}$$

$$\Delta[A_j \cdot P(x, t) - \frac{1}{2}\frac{\partial}{\partial x_i} B_{ij} \cdot P(x, t)] = 0 \qquad \text{(B9)}$$

From eqns (B8) and (B9) follows the matching conditions eqns (106) and (107) accordingly.

We now represent $P(x, t)$ inside the domains $\{I\}$ and $\{II\}$ separated by the conditions $x_1 = c_0$, in the form of integrals of the boundary functions P_1 and P_2.

Remark B1. The integral form of solution for eqn (110) in domain $\{I\}$ defines as a fundamental solution $q_1(x_1, x_2, t, \eta, \xi, \tau)$ of eqn (110), continued from this domain $\{I\}$ onto the entire phase plane (x_1, x_2) of the system eqn (108). In this case we use the theorem[17] that the fundamental solution $q_1(x_1, x_2, t, \eta, \xi, \tau)$ of eqn (110), considered on the entire phase plane of eqn (108), coincides with probability density function of any random Markovian process.

We continue eqn (110) from the domain $\{I\}$ onto the entire phase plane (x_1, x_2) of eqn (108). Its adjoint equation has the form:

$$N[q(\eta, \xi, \tau)] = \frac{\partial q}{\partial \tau} + \xi\frac{\partial q}{\partial \eta} + [-2\alpha\xi - \omega_0^2\eta]\frac{\partial q}{\partial \xi} + \frac{C_f^2}{2}\cdot\frac{\partial^2 q}{\partial \xi^2}\bigg|_{0 \le x_1 \le c_0} \qquad \text{(B10)}$$

The fundamental solution of eqn (110) is the transition probability density function $q_1(x_1, x_2, t, \eta, \xi, \tau)$ for a Markovian process satisfying eqn (108) under the condition $x \in \{I\}$, where $q_1(x_1, x_2, t, \eta, \xi, \tau) \to \delta(x_1, - \eta, x_2 - \xi)$ as $(t - \tau) \to 0$. It is also known[2,17] that $q(\eta, \xi, \tau)$ as a function of the three arguments satisfies eqn (B10). The process in the linear eqn (108) is Gaussian, hence q_1 is completely determined by the vector expectation $M[x_1, x_2, t, \xi, \tau] = (m_1, m_2)$ and the correlation matrix $\|\mathbf{D}_{ij}\|$

$$q_1(x_1, x_2, \eta, \xi, \tau) = \frac{1}{2\pi\sqrt{|d|}} \times \exp\left\{-\frac{1}{2|d|}\sum_{i,j=1}^{2} \mathbf{D}_{ij}(x_i - m_i)(x_j - m_j)\right\} \qquad \text{(B11)}$$

Here $|d|$ is the determinant of the correlation matrix, and $\mathbf{D}_{ij}$ is the signed minor of the element d_{ij} in that matrix. The vector (m_1, m_2) is found as the solution of the system:

$$\dot{m}_1 = m_2;\ \dot{m}_2 = -2\alpha m_2 - \omega_i^2 m_1\ (i = 1, 2, \ldots, n) \qquad \text{(B12)}$$

with initial conditions $m_1(\tau) = \eta, m_2(\tau) = \xi$.

The elements of the correlation matrix are found from the system:[17,24]

$$\dot{d}_{11} = 2d_{12};\ \dot{d}_{12} = d_{22} - \omega_i^2 d_{11} - 2\alpha d_{12};$$

$$\dot{d}_{22} = -2(\omega_i^2 d_{12} - 4\alpha d_{22} + C_f^2;\ d_{12} = d_{21} \qquad \text{(B13)}$$

with initial conditions $d_{ij}\ (t = \tau) = 0$ for $i, j = 1, 2$.

For the solution of problem, the desired representation of the function $P(x, t)$ in domain $\{I\}$ in terms of the boundary functions P_1 and P_2 we use the following method.

Let us examine the equation:

$$\int_0^{t-\sigma} d\tau \iint_{\{(\eta,\xi):|\eta|<c_i, 0\le\xi\le\infty\}} \{P(\eta, \xi, \tau)N[q_1(\eta, \xi, \tau)] - q_1(\eta, \xi, \tau)L[P(\eta, \xi, \tau)]\}\, d\eta\, d\xi$$
$$\equiv \int_0^{t-\sigma} d\tau \iint_{\{(\eta,\xi):|\eta|<c_i, 0\le\xi\le\infty\}} \left\{\frac{\partial}{\partial \tau}[q_1 P] + \frac{\partial}{\partial \eta}[\xi P q_1] + \frac{\partial}{\partial \xi}\left[(2\alpha + \omega_0^2\eta)q_1 P + \frac{C_f^2}{2}\frac{\partial q_1}{\partial \xi} - \frac{C_f^2}{2} q_1 \frac{\partial P}{\partial \xi}\right]\right\} d\eta\, d\xi$$
$$= 0 \qquad \text{(B14)}$$

According to the Gauss–Ostrogradsky formula, the left-hand side of eqn () is equal to an integral of the second kind over the surface bounding the following volume: $0 \le \tau \le t - \sigma$, $0 \le \xi \le l$, $0 \le \eta \le c_i$. In passing to limit $l \to \infty$, $\sigma \to 0$ is essential [to take account of the rapid decay of the functions $P(x,t)$ and q_1 as $|\eta| \to \infty$ and $|\xi| \to \infty$], so that all integrals encountered converge, obtain the representation of function $P(x, t) \in \{I\}$ through the boundary functions P_1 and P_2. In this case, we note that if $x \in \{I\}$, then:

$$\lim_{\sigma \to \infty} \iint_{\{(\eta,\xi):|\eta|<c_0\}} q_1(x_1, x_2, \eta, \xi, t - \sigma)P(\eta, \xi, t - \sigma)\, d\eta\, d\xi = P(x_1, x_2, t) \qquad \text{(B15)}$$

Now, denoting $P(x_1, x_2, t) = f(x_1, x_2)$ we can write from eqn (B15) for domain $0 \le x_1 < c_0, 0 \le x_2 < \infty$ eqn (121).

In domain $\{II\}$ of the phase space (x_1, x_2), eqn (111) is satisfied by the Gaussian transition probability density function $q_2(x_1, x_2, t, \xi, \tau)$ the parameters of which are determined from eqns (B12) and (B13) with replaced $\omega_0^2 \to \omega_1^2$ and $\omega_0^2 > \omega_1^2$. Taking account of the boundary conditions of eqn (112) and eqn (115), we find the representation of $P(x, t)$ in the domain $\{II\}$ has the form of eqn (122).

Remark B2. For eqn (131) with the variable structure we can write:

$$N[q_1(\eta, \xi, \tau)] = \frac{\partial q}{\partial \tau} + \xi \frac{\partial q}{\partial \eta} + [-2\alpha\xi - \omega_0^2 + \Phi_1(\tau)]\frac{\partial q}{\partial \xi} + \frac{C_f^2}{2}\frac{\partial^2 q}{\partial \xi^2} = 0 \tag{B16}$$

and

$$q_1(x_1, x_2, t, \eta, \xi, \tau) = \frac{1}{2\pi\sqrt{|d|}} \exp\left\{ -\frac{1}{2|d|}\sum_{i,j=1}^{2} \mathbf{D}_{ij}(x_i - m_i)(x_j - m_j) \right\} \tag{B17}$$

The parameters m_1, m_2 and $\mathbf{D}_{ij}$ we define from eqns (B12) and (B13). Eqn (B14) for this case writes as

$$\int_0^{t-\sigma} d\tau \int\int_{\{(\eta,\xi):(c_1\eta + c_2\xi)>0\}} \times \{P(\eta, \xi, \tau)N[q_1(\eta, \xi, \tau)] - q_1(\eta, \xi, \tau)L[P(\eta, \xi, \tau)]\} \times d\eta\, d\xi \equiv \int_0^{t-\sigma} d\tau \int\int_{\{(\eta,\xi):(c_1\eta + c_2\xi)>0\}}$$
$$\times \left\{ \frac{\partial}{\partial \tau}[q_1 P] + \frac{\partial}{\partial \eta}[\xi P q_1] + \frac{\partial}{\partial \xi} \times \left[(2\alpha\xi + \omega_0^2\eta - \Phi_1(\tau))q_1 P + \frac{C_f^2}{2}\frac{\partial q_1}{\partial \xi} - \frac{C_f^2}{2} q_1 \frac{\partial P}{\partial \xi} \right] \right\} d\eta\, d\xi = 0 \tag{B18}$$

and

$$\lim_{\sigma \to \infty} \int\int_{\{(\eta,\xi):(c_1\eta + c_2\xi)>0\}} q_1(x_1, x_2, t, \eta, \xi, t - \sigma) \times P(\eta, \xi, t - \sigma)\, d\eta\, d\xi = P(x_1, x_2, t) \tag{B19}$$

In this case we obtain the results.[17,24]

Remark B3. The kernels of eqns (140) and (141) define as Gaussian probability density functions q_1 and q_2 that rapid decrease as $|x_1, x_2| \to \infty$. In this case, these kernel functions can be represent as Chebyshev–Hermitian expansion and coefficients of this expansion has the function form an expectations (m_1, m_2) and correlation matrix $\mathbf{D}_{ij}(t, \tau)$. With the approximation q_1 and q_2 as partial sums of series we obtain the linear integral equations with degenerate kernels. The solution of these equations reduce to the solution of algebraic equations and can be solvable, for example, by successive approximation methods or methods of steepest descent. The accuracy of this solution can be evaluate with well-known formulas of linear integral equation theory.